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Algebras with separating almost cyclic coherent Auslander–Reiten components ☆

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Received 1 August 2004

Available online 17 May 2005

Communicated by Kent R. Fuller

Dedicated to Claus Michael Ringel on the occasion of his sixtieth birthday

Abstract

We prove that the class of Artin algebras whose Auslander–Reiten quiver admits a separating family of almost cyclic coherent components coincides with the class of generalized multicoil enlargements of concealed canonical algebras. Moreover, the module category, homological properties and the representation type of Artin algebras with separating families of almost cyclic coherent Auslander–Reiten components are described.

1. Introduction and the main results

Throughout the paper by an algebra we mean a basic Artin algebra over a fixed commutative Artin ring R. For an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated right A-modules, and by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ consisting of indecomposable modules. We shall denote by $\operatorname{rad}(\operatorname{mod} A)$ the *Jacobson radical* of $\operatorname{mod} A$, and by $\operatorname{rad}^{\infty}(\operatorname{mod} A)$ the intersection of all powers $\operatorname{rad}^{i}(\operatorname{mod} A)$, $i \geqslant 1$, of $\operatorname{rad}(\operatorname{mod} A)$. Moreover,

☆ The authors gratefully acknowledge support from the Polish Scientific Grant KBN No. 5 P03A 008 21.
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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2005.03.021

we denote by Γ_A the Auslander–Reiten quiver of A, and by τ_A and τ_A^- the Auslander–Reiten translations $D\,\mathrm{Tr}$ and $\mathrm{Tr}\,D$, respectively [1]. We will not distinguish between an indecomposable A-module, its isomorphism class, and the vertex of Γ_A corresponding to it. Following [38] a family $\mathcal{C}$ of components of Γ_A is said to be *generalized standard* if $\mathrm{rad}^\infty(X,Y)=0$ for all modules X and Y from $\mathcal{C}$. We note that then different components in a generalized standard family $\mathcal{C}$ are orthogonal. Recall also that the family $\mathcal{C}$ is called *sincere* if any simple A-module occurs as a composition factor of a module in $\mathcal{C}$.

The Auslander–Reiten quiver is an important combinatorial and homological invariant of the module category $\mathrm{mod}\,A$ of an algebra A. Frequently, we may recover A from the behavior of distinguished components of Γ_A in the category $\mathrm{mod}\,A$. For example, it is the case for the tilted algebras [16,23,37], or more generally, the double tilted [31] and the generalized double tilted algebras [32,44] whose Auslander–Reiten quiver admits a faithful component with a finite section (respectively double section, multisection) satisfying a vanishing hom-condition.

In the representation theory of algebras a prominent role is played by the algebras with a separating family of stable tubes (in the sense of Ringel [33]). The class of these algebras contains the tame hereditary algebras [10,11,33], the tame concealed algebras [34], the tubular algebras [18,34], the canonical algebras [14,20,34,35] and, more generally, the concealed canonical algebras [19]. It has been proved in [21] (see also [40]) that the class of algebras with a sincere separating family of stable tubes coincides with the class of concealed canonical algebras. This was deepened in [30,43], where a characterization of concealed canonical algebras in terms of external short paths (cycles) has been established. In order to deal with wider classes of algebras, we need a slightly more general concept of a separating family of components. Namely, a family $\mathcal{C}=(\mathcal{C}_i)_{i\in I}$ of components of Γ_A is said to be *separating* in $\mathrm{mod}\,A$ if the modules in $\mathrm{ind}\,A$ split into three disjoint classes $\mathcal{P}_A$, $\mathcal{C}_A=\mathcal{C}$ and $\mathcal{Q}_A$ such that:

(S1) $\mathcal{C}_A$ is a sincere generalized standard family of components;
(S2) $\mathrm{Hom}_A(\mathcal{Q}_A,\mathcal{P}_A)=0$, $\mathrm{Hom}_A(\mathcal{Q}_A,\mathcal{C}_A)=0$, $\mathrm{Hom}_A(\mathcal{C}_A,\mathcal{P}_A)=0$;
(S3) any morphism from $\mathcal{P}_A$ to $\mathcal{Q}_A$ factors through $\mathrm{add}\,\mathcal{C}_A$.

We then say that $\mathcal{C}_A$ separates $\mathcal{P}_A$ from $\mathcal{Q}_A$ and write $\mathrm{ind}\,A=\mathcal{P}_A\vee\mathcal{C}_A\vee\mathcal{Q}_A$. We also note that then $\mathcal{P}_A$ and $\mathcal{Q}_A$ are uniquely determined by $\mathcal{C}_A$ (see [4, (2.1)] or [34, (3.1)]). Recall from [15] that an algebra A is called *quasitilted* if A has global dimension at most two and every module in $\mathrm{ind}\,A$ has projective or injective dimension at most one. It has been recently proved in [17] that the class of quasitilted algebras consists of the tilted algebras (endomorphism algebras of tilting modules over hereditary algebras) and the quasitilted algebras of canonical type (endomorphism algebras of tilting modules over canonical algebras). Moreover, by [22, Theorem 3.4], an algebra A is quasitilted of canonical type if and only if Γ_A admits a separating family of semiregular tubes.

The main aim of the paper is to describe the structure and homological properties of algebras whose Auslander–Reiten quiver admits a separating family of almost cyclic coherent components. A component Γ of Γ_A is said to be *almost cyclic* if all but finitely many modules of Γ lie on oriented cycles contained entirely in Γ. Further, a component Γ of Γ_A is said to be *coherent* if the following two conditions are satisfied:

(C1) For each projective module P in Γ there is an infinite sectional path $P = X_1 \to X_2 \to \cdots \to X_i \to X_{i+1} \to X_{i+2} \to \cdots$ (that is, $X_i \neq \tau_A X_{i+2}$ for any $i \geqslant 1$) in Γ.
(C2) For each injective module I in Γ there is an infinite sectional path $\cdots \to Y_{j+2} \to Y_{j+1} \to Y_j \to \cdots \to Y_2 \to Y_1 = I$ (that is, $Y_{j+2} \neq \tau_A Y_j$ for any $j \geqslant 1$) in Γ.

It has been proved in [25, Theorem A] that a component Γ of Γ_A is almost cyclic and coherent if and only if Γ is a generalized multicoil, that is, can be obtained from a finite family of stable tubes by a sequence of admissible operations (see Section 3 for details). We introduce (Section 3) the concept of a generalized multicoil enlargement of a (not necessarily connected) concealed canonical algebra, extending the concept of a coil enlargement of a concealed canonical algebra introduced in [4]. Following [30] by an *external short path* in $\operatorname{ind} A$, with respect to a family $\mathcal{C}$ of components of Γ_A, we mean a sequence $X \to Y \to Z$ of nonzero nonisomorphisms in $\operatorname{ind} A$ such that the modules X and Z belong to $\mathcal{C}$ but Y is not in $\mathcal{C}$.

The following theorem is the first main result of the paper.

Theorem A. *Let A be an algebra. The following statements are equivalent*:

(i) *Γ_A admits a separating family of almost cyclic coherent components.*
(ii) *Γ_A admits a sincere generalized standard family of almost cyclic coherent components without external short paths.*
(iii) *A is a generalized multicoil enlargement of a concealed canonical algebra C.*

The concealed canonical algebra C is called the *core* of A and the number $m = m(A)$ of connected summands of C is a numerical invariant of A. We note that $m(A)$ can be arbitrary large, even if A is connected.

As an immediate consequence of Theorem A and [21, Theorem 1.1] we obtain the following fact.

Corollary B. *Let A be an algebra with a separating family of almost cyclic coherent components in Γ_A. Then the ordinary valued quiver of A has no oriented cycles. Moreover, if A is connected, then the centre of A is a field.*

The second main result of the paper describes the structure of the module category of an algebra with a separating family of almost cyclic coherent Auslander–Reiten components.

Theorem C. *Let A be an algebra with a separating family $\mathcal{C}_A$ of almost cyclic coherent components in Γ_A, and* $\operatorname{ind} A = \mathcal{P}_A \vee \mathcal{C}_A \vee \mathcal{Q}_A$. *Then the following statements hold.*

(i) *There is a unique factor algebra A_l of A which is a quasitilted algebra of canonical type having a separating family $\mathcal{T}_{A_l}$ of coray tubes such that* $\operatorname{ind} A_l = \mathcal{P}_{A_l} \vee \mathcal{T}_{A_l} \vee \mathcal{Q}_{A_l}$ *and $\mathcal{P}_A = \mathcal{P}_{A_l}$ consists of all proper predecessors of $\mathcal{C}_A$ in* $\operatorname{ind} A$.
(ii) *There is a unique factor algebra A_r of A which is a quasitilted algebra of canonical type having a separating family $\mathcal{T}_{A_r}$ of ray tubes such that* $\operatorname{ind} A_r = \mathcal{P}_{A_r} \vee \mathcal{T}_{A_r} \vee \mathcal{Q}_{A_r}$ *and $\mathcal{Q}_A = \mathcal{Q}_{A_r}$ consists of all proper successors of $\mathcal{C}_A$ in* $\operatorname{ind} A$.

The algebra A_l (respectively A_r) is said to be the *left* (respectively *right*) *quasitilted algebra* of A. We note that in general the algebras A_l and A_r are not connected, even if A is connected. In fact, the both A_l and A_r have exactly $m(A)$ connected algebra summands.

The known structure of the Auslander–Reiten quivers of quasitilted algebras of canonical type (see [6,13,22,27,41]) and Theorem C lead to the following corollary.

Corollary D. *Let A be an algebra with a separating family $\mathcal{C}_A$ of almost cyclic coherent components in Γ_A, and* $\operatorname{ind} A = \mathcal{P}_A \vee \mathcal{C}_A \vee \mathcal{Q}_A$. *Then the following statements hold.*

(i) *Every component of Γ_A not in $\mathcal{C}_A$ lies entirely in $\mathcal{P}_A$ or lies entirely in $\mathcal{Q}_A$.*
(ii) *Every component of Γ_A contained in $\mathcal{P}_A$ is either preprojective, a stable tube $\mathbb{Z}\mathbb{A}_\infty/(\tau^n)$, for some $n \geqslant 1$, of the form $\mathbb{Z}\mathbb{A}_\infty$, or can be obtained from a stable tube or a component of type $\mathbb{Z}\mathbb{A}_\infty$ by a finite number of ray insertions.*
(iii) *Every component of Γ_A contained in $\mathcal{Q}_A$ is either preinjective, a stable tube $\mathbb{Z}\mathbb{A}_\infty/(\tau^n)$, for some $n \geqslant 1$, of the form $\mathbb{Z}\mathbb{A}_\infty$, or can be obtained from a stable tube or a component of type $\mathbb{Z}\mathbb{A}_\infty$ by a finite number of coray insertions.*
(iv) *Γ_A admits exactly $m(A)$ preprojective components and exactly $m(A)$ preinjective components.*

The third main result describes the homological properties of algebras with separating families of almost cyclic coherent Auslander–Reiten components.

Theorem E. *Let A be an algebra with a separating family $\mathcal{C}_A$ of almost cyclic coherent components in Γ_A, and* $\operatorname{ind} A = \mathcal{P}_A \vee \mathcal{C}_A \vee \mathcal{Q}_A$. *Then the following statements hold.*

(i) $\operatorname{pd}_A X \leqslant 1$ *for any module X in $\mathcal{P}_A$.*
(ii) $\operatorname{id}_A X \leqslant 1$ *for any module X in $\mathcal{Q}_A$.*
(iii) $\operatorname{pd}_A X \leqslant 2$ *and* $\operatorname{id}_A X \leqslant 2$ *for any module X in $\mathcal{C}_A$.*
(iv) $\operatorname{gl\,dim} A \leqslant 3$.

Finally, we will describe the representation type of algebras with separating families of almost cyclic coherent Auslander–Reiten components. Following [8] an algebra A is said to be *strictly wild* if there are A-modules X and Y whose endomorphism rings $\operatorname{End}_A(X)$ and $\operatorname{End}_A(Y)$ are division rings, $\operatorname{Hom}_A(X,Y) = 0 = \operatorname{Hom}_A(Y,X)$ and $\dim_{\operatorname{End}_A(X)} \operatorname{Ext}^1_A(X,Y) \cdot \dim_{\operatorname{End}_A(Y)} \operatorname{Ext}^1_A(X,Y) \geqslant 5$.

If R is a field k, then it follows from [33] that A is strictly wild if and only if there is a finite field extension K of k and an $K\langle x,y\rangle$-A-bimodule M which is finitely generated projective over $K\langle x,y\rangle$ and such that the tensor product functor $\otimes_{K\langle x,y\rangle} M : \operatorname{Mod} K\langle x,y\rangle \to \operatorname{Mod} A$ is fully faithful. Here, $K\langle x,y\rangle$ denotes the free associative K-algebra in two generators, and $\operatorname{Mod} K\langle x,y\rangle$ and $\operatorname{Mod} A$ the categories of all $K\langle x,y\rangle$-modules and all A-modules, respectively. Moreover, we say that an algebra A is *wild* if there exists a field K and an $K\langle x,y\rangle$-A-bimodule M, finitely generated and projective as $K\langle x,y\rangle$-module, such that the functor $\otimes_{K\langle x,y\rangle} M : \operatorname{Mod} K\langle x,y\rangle \to \operatorname{Mod} A$ preserves indecomposability and isomorphism classes of modules (see [8]). It is known that a wild hereditary algebra is strictly wild [8,33] but in general the converse is not true.

Assume now that R is an algebraically closed field k. Then following [12], an algebra A is said to be *tame* if, for each dimension d, there exists a finite number of $k[x]$-A-bimodules $M_1, \dots, M_{n(d)}$ which are finitely generated and free as left $k[x]$-modules and all but a finite number of isomorphism classes of indecomposable A-modules of dimension d are of the form $k[x]/(x-\lambda) \otimes_{k[x]} M_i$ for some $\lambda \in k$ and some $1 \leqslant i \leqslant n(d)$. Then the remarkable theorem of Drozd asserts that every algebra is either tame or wild. Recall also that a basic algebra A over an algebraically closed field k can be presented as a bound quiver algebra $A = kQ/I$ where $Q = Q_A$ is the ordinary quiver of A and I is an admissible ideal of the path algebra kQ of Q. If Q_A has no oriented cycles then we may define the *Tits form* q_A (integral quadratic) on the Grothendieck group $K_0(A) = \mathbb{Z}^{Q_0}$ of A as follows:

$$q_A(x) = \sum_{i \in Q_0} x_i^2 - \sum_{(i \to j) \in Q_1} x_i x_j + \sum_{i,j \in Q_0} r_{ij} x_i x_j,$$

where $x = (x_i)_{i \in Q_0} \in K_0(A)$, Q_0 and Q_1 are the sets of vertices and arrows of Q, respectively, and r_{ij} is the number of relations from i to j in a minimal set of relations generating the ideal I (see [5]). It is known (see [28]) that if A is tame then q_A is *weakly nonnegative*, that is, $q_A(x) \geqslant 0$ for any $x \in \mathbb{N}^{Q_0}$.

It has been shown in [21, Theorem 7.1] that the representation type of a concealed canonical algebra C is controlled by its genus $g(C)$ determined by the ranks of tubes of its separating tubular family. This can be extended to arbitrary quasitilted algebras (see Section 2 for details). Then the genus $g(A)$ of an algebra A with a separating family of almost cyclic coherent components in Γ_A is defined to be the maximum of the genus $g(A_l)$ and $g(A_r)$ of its left and right quasitilted algebras A_l and A_r.

The following theorem is the final main result of the paper.

Theorem F. *Let A be an algebra with a separating family of almost cyclic coherent components in Γ_A. The following statements are equivalent*:

(i) *A is not strictly wild.*
(ii) *A is not wild.*
(iii) $g(A) \leqslant 1$.
(iv) *A_l and A_r are products of tilted algebras of Euclidean type or tubular algebras.*

Moreover, if R is an algebraically closed field then the above statements are equivalent to each of the two statements below.

(iv) *A is tame.*
(v) *The Tits form q_A is weakly nonnegative.*

2. Quasitilted algebras of canonical type

Throughout the paper by a canonical algebra we mean a product of a finite number of connected canonical algebras over a field in the sense of Ringel [35]. It has been proved in

[35] that, if Λ is a canonical algebra, then $\operatorname{ind}\Lambda = \mathcal{P}_\Lambda \vee \mathcal{T}_\Lambda \vee \mathcal{Q}_\Lambda$ for a family $\mathcal{T}_\Lambda$ of stable tubes of Γ_Λ separating $\mathcal{P}_\Lambda$ from $\mathcal{Q}_\Lambda$. Following [19], an algebra C is called *concealed canonical* if C is the endomorphism algebra $\operatorname{End}_\Lambda(T)$, for some canonical algebra Λ and a tilting Λ-module T whose indecomposable direct summands belong to $\mathcal{P}_\Lambda$. Then the images of modules from $\mathcal{T}_\Lambda$ via the functor $\operatorname{Hom}_\Lambda(T,-)$ form a separating family $\mathcal{T}_C$ of stable tubes of Γ_C, and in particular we have a decomposition $\operatorname{ind} C = \mathcal{P}_C \vee \mathcal{T}_C \vee \mathcal{Q}_C$ of the category $\operatorname{ind} C$. It has been proved by Lenzing and de la Peña [21, Theorem 1.1] that the class of concealed canonical algebras coincides with the class of all algebras with a separating family of stable tubes. We also note that if C is connected then the index set $\mathbb{X}$ of a separating tubular family $\mathcal{T}_C = (\mathcal{T}_x)_{x\in\mathbb{X}}$ of Γ_C is in a natural bijection with the set of regular components of a tame hereditary algebra $\left[\begin{smallmatrix} F & M \\ 0 & G \end{smallmatrix}\right]$, where F and G are finite central skew field extensions of a field k and the F–G-bimodule M satisfies $\dim_F M \cdot \dim M_G = 4$ (see [7,33,35]). Moreover, if R is an algebraically closed field, then $\mathbb{X}$ is in a natural bijection with the projective line over k, and is equipped with the structure of a weighted projective line [14]. We refer to [14,18–21,27,33,34] for the representation theory of canonical and concealed algebras.

We will need also the following characterization of concealed canonical algebras.

Theorem 2.1. *An algebra A is a concealed canonical algebra if and only if Γ_A has a sincere family of pairwise orthogonal stable tubes without external short paths.*

Proof. For finite-dimensional algebras over an algebraically closed field, the theorem is proved in [43, Theorem 1.6] (see also [30, Theorem 3.1]). The proof works also in the general case due to the characterization of quasitilted algebras over arbitrary fields established in [17], and general arguments applied in the proofs of [22, Theorem 3.4], [30, Corollary 1.6], and [42, Proposition 1.1]. □

An algebra A is said to *quasitilted of canonical type* (respectively *almost concealed canonical*) if A is the endomorphism algebra $\operatorname{End}_\Lambda(T)$, for some canonical algebra Λ and a tilting Λ-module T (respectively a tilting Λ-module whose indecomposable direct summands belong to $\mathcal{P}_\Lambda \vee \mathcal{T}_\Lambda$). It has been proved in [22, Theorem 3.4] that A is quasitilted if and only if Γ_A admits a separating family of semiregular tubes. Moreover, the class of almost concealed algebras coincides with the class of tubular extensions of concealed canonical algebras, and with the class of algebras having a separating family of tubes without injective modules (ray tubes) (see [19, Theorem 3.1] and [22, Theorem 3.4]).

Let Λ be a connected canonical algebra and $\mathcal{T}_\Lambda$ a separating family of stable tubes of Γ_Λ. Then, by [35], all but finitely many tubes in $\mathcal{T}_\Lambda$ are homogeneous (of rank 1). In fact, it is known that all tubes of $\mathcal{T}_\Lambda$ are homogeneous if and only if Λ is a tame hereditary algebra of the form $\left[\begin{smallmatrix} F & M \\ 0 & G \end{smallmatrix}\right]$ described above. In such a case the genus $g(\Lambda)$ of Λ is defined to be 0. Assume now that Γ_Λ admits a nonhomogeneous tube, and denote by $p_1, p_2, \dots, p_t$ the ranks of all nonhomogeneous tubes in $\mathcal{T}_\Lambda$. Then the genus $g(\Lambda)$ of Λ is defined as

$$g(\Lambda) = 1 + \frac{1}{2}\left((t-2)p - \sum_{i=1}^{t} \frac{p}{p_i}\right),$$

where $p = \operatorname{lcm}(p_1, p_2, \dots, p_t)$. For a tilting module T from $\operatorname{add}(\mathcal{P}_\Lambda \vee \mathcal{T}_\Lambda)$ and the associated almost concealed canonical algebra $A = \operatorname{End}_\Lambda(T)$, we define the genus $g(A)$ of A to be the genus $g(\Lambda)$ of Λ. Then we have the following consequence of [19, Theorem 3.1], [21, Theorem 7.1] and [22, Theorem 3.4].

Theorem 2.2. *Let A be a connected almost concealed canonical algebra. Then the following statements hold.*

(i) $g(A) < 1$ *if and only if A is a tilted algebra of Euclidean type having a complete section in the preinjective component.*
(ii) $g(A) = 1$ *if and only if A is tame of tubular type.*
(iii) $g(A) > 1$ *if and only if A is wild.*

3. Almost cyclic coherent generalized standard components

Recall from [9,34] that a translation quiver Γ is called a *tube* if it contains a cyclical path and if its underlying topological space is homeomorphic to $S^1 \times \mathbb{R}^+$ (where S^1 is the unit circle, and $\mathbb{R}^+$ the nonnegative real line). A tube has only two types of arrows: arrows pointing to infinity and arrows pointing to the mouth. Tubes containing neither projective vertices nor injective vertices are called *stable*. Recall that if $\mathbb{A}_\infty$ is the quiver $0 \to 1 \to 2 \to \cdots$ (with the trivial valuations $(1,1)$), then $\mathbb{Z}\mathbb{A}_\infty$ is the translation quiver of the form:

$$\begin{tikzcd}[column sep=small]
 & (i-1,0) \arrow[dr] & & (i,0) \arrow[dr] & & (i+1,0) \arrow[dr] & & (i+2,0) \arrow[dr] & \\
\iddots \arrow[ur] & & (i-1,1) \arrow[ur] \arrow[dr] & & (i,1) \arrow[ur] \arrow[dr] & & (i+1,1) \arrow[ur] \arrow[dr] & & \ddots \\
 & \iddots \arrow[ur] & & (i-1,2) \arrow[ur] \arrow[dr] & & (i,2) \arrow[ur] \arrow[dr] & & \ddots & \\
 & & \iddots \arrow[ur] & & \ddots \ \iddots \arrow[ur] & & \ddots & &
\end{tikzcd}$$

with $\tau(i,j) = (i-1,j)$ for $i \in \mathbb{Z}$, $j \in \mathbb{N}$. For $r \geqslant 1$, denote by $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ the translation quiver Γ obtained from $\mathbb{Z}\mathbb{A}_\infty$ by identifying each vertex (i,j) of $\mathbb{Z}\mathbb{A}_\infty$ with the vertex $\tau^r(i,j)$ and each arrow $x \to y$ in $\mathbb{Z}\mathbb{A}_\infty$ with the arrow $\tau^r x \to \tau^r y$. The translation quivers of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, $r \geqslant 1$, are called *stable tubes of rank r*. The *rank* of a stable tube Γ is the least positive integer r such that $\tau^r x = x$ for all x in Γ. A stable tube of rank 1 is said to be *homogeneous*. The τ-orbit of a stable tube Γ formed by all vertices having exactly one predecessor is said to be the *mouth* of Γ.

The following characterization of generalized standard stable tubes of an Auslander–Reiten quiver has been established in [38, Corollary 5.3] (see also [39, Lemma 3.1]).

Proposition 3.1. *Let A be an algebra and Γ a stable tube of Γ_A. The following statements are equivalent*:

(i) *Γ is generalized standard.*
(ii) *The mouth of Γ consists of pairwise orthogonal bricks.*
(iii) $\operatorname{rad}^\infty(X,X) = 0$ *for any module X in Γ.*

An indecomposable module X is called a *brick* if its endomorphism algebra $F_X = \mathrm{End}_A(X)$ is a division algebra. We also note that the division algebras F_X of all modules X lying on the mouth of a generalized standard stable tube of Γ_A are isomorphic.

We also note that the generalized canonical algebras (introduced in [43]) provide a wide class of algebras whose Auslander–Reiten quivers admit generalized standard stable tubes.

It has been proved in [25, Theorem A] that a connected component Γ of Γ_A is almost cyclic and coherent if and only if Γ is a generalized multicoil, obtained from a family of stable tubes by a sequence of operations called admissible. Our task in this section is to recall the letter and simultaneously define the corresponding enlargements of algebras. We start with the concepts of one-point extensions and one-point coextensions of algebras. Let A be an algebra, let F be a division algebra over R, and let $M = {}_F M_A$ be an F–A-bimodule such that $M_A \in \mathrm{mod}\, A$ and R acts centrally on ${}_F M_A$. Then the *one-point extension* of A by M is the matrix Artin R-algebra of the form

$$A[M] = \begin{bmatrix} A & 0 \\ {}_F M_A & F \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ m & f \end{bmatrix};\ f \in F,\ a \in A,\ m \in M \right\}$$

with the usual addition and multiplication. Then the valued quiver $Q_{A[M]}$ of $A[M]$ contains the valued quiver Q_A of A as a convex subquiver, and there is an additional (extension) vertex which is a source. We may identify the category $\mathrm{mod}\, A[M]$ with the category whose objects are triples (V, X, φ), where $X \in \mathrm{mod}\, A$, $V \in \mathrm{mod}\, F$, and $\varphi : V_F \to \mathrm{Hom}_A(M, X)_F$ is an F-linear map. A-morphism $h : (V, X, \varphi) \to (W, Y, \psi)$ is given by a pair (f, g), where $f : V \to W$ is F-linear, $g : X \to Y$ is a morphism in $\mathrm{mod}\, A$ and $\psi f = \mathrm{Hom}_A(M, g)\varphi$. Then the new indecomposable projective $A[M]$-module P is given by the triple $(F, M, \bullet)$ where $\bullet : F_F \to \mathrm{Hom}_A(M, M)_F$ assigns to the identity element of F the identity morphism of M. An important class of such one-point extensions occurs in the following situation. Let Λ be a basic Artin R-algebra, P an indecomposable projective Λ-module, ${}_\Lambda \Lambda = P \oplus Q$, and assume that $\mathrm{Hom}_\Lambda(P, Q \oplus \mathrm{rad}\, P) = 0$. Since P is indecomposable projective, $S = P/\mathrm{rad}\, P$ is a simple Λ-module and hence $\mathrm{End}_\Lambda(S)$ is a division algebra. Moreover, the canonical homomorphism of algebras $\mathrm{End}_\Lambda(P) \to \mathrm{End}_\Lambda(S)$ is an isomorphism. Then we obtain isomorphisms of algebras

$$\Lambda \cong \mathrm{End}_\Lambda(\Lambda_\Lambda) \cong \begin{bmatrix} A & 0 \\ {}_F M_A & F \end{bmatrix} = A[M],$$

where $F = \mathrm{End}_\Lambda(P)$, $A = \mathrm{End}_\Lambda(Q)$, and $M = {}_F M_A = \mathrm{Hom}_\Lambda(Q, P) \cong \mathrm{rad}\, P$. Clearly R acts centrally on ${}_F M_A$. We note that if the valued quiver of an Artin algebra Λ has no oriented cycles then Λ can be obtained from a semisimple algebra by a sequence of one-point extensions of the above form. Dually, one defines also the *one-point coextension* of A by ${}_F M_A$ as the matrix algebra

$$[M]A = \begin{bmatrix} F & 0 \\ D({}_F M_A) & A \end{bmatrix}.$$

For each bimodule ${}_F M_A$ considered in the paper we assume that A is an algebra, $M_A \in \mathrm{mod}\, A$, F is a division algebra, and R acts centrally on ${}_F M_A$.

For a division algebra F and $r \geqslant 1$, we denote by $T_r(F)$ the $r \times r$-lower triangular matrix algebra

$$\begin{bmatrix} F & 0 & 0 & \dots & 0 & 0 \\ F & F & 0 & \dots & 0 & 0 \\ F & F & F & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ F & F & F & \dots & F & 0 \\ F & F & F & \dots & F & F \end{bmatrix}.$$

Given a generalized standard component Γ of Γ_A, and an indecomposable module X in Γ, the *support* $\mathcal{S}(X)$ of the functor $\mathrm{Hom}_A(X,-)|_\Gamma$ is the R-linear category defined as follows [3]. Let $\mathcal{H}_X$ denote the full subcategory of Γ consisting of the indecomposable modules M in Γ such that $\mathrm{Hom}_A(X,M) \neq 0$, and $\mathcal{I}_X$ denote the ideal of $\mathcal{H}_X$ consisting of the morphisms $f : M \to N$ (with M, N in $\mathcal{H}_X$) such that $\mathrm{Hom}_A(X,f) = 0$. We define $\mathcal{S}(X)$ to be the quotient category $\mathcal{H}_X/\mathcal{I}_X$. Following the above convention, we usually identify the R-linear category $\mathcal{S}(X)$ with its quiver.

From now on let A be an algebra and Γ be a family of generalized standard infinite components of Γ_A. For an indecomposable brick X in Γ, called the *pivot*, one defines five admissible operations (ad 1)–(ad 5) and their dual (ad 1*)–(ad 5*) modifying the translation quiver $\Gamma = (\Gamma, \tau)$ to a new translation quiver (Γ', τ') and the algebra A to a new algebra A', depending on the shape of the support $\mathcal{S}(X)$ (see [25, Section 2] for the figures illustrating the modified translation quivers Γ'). Let $F = F_X = \mathrm{End}_A(X)$ be the division algebra associated to X.

(ad 1) Assume $\mathcal{S}(X)$ consists of an infinite sectional path starting at X:

$$X = X_0 \to X_1 \to X_2 \to \cdots$$

In this case, we let $t \geqslant 1$ be a positive integer, $D = T_t(F)$ and $Y_1, Y_2, \dots, Y_t$ denote the indecomposable injective D-modules with $Y = Y_1$ the unique indecomposable projective–injective D-module. We define the *modified algebra* A' of A to be the one-point extension

$$A' = (A \times D)[X \oplus Y]$$

and the *modified translation quiver* Γ' of Γ to be obtained by inserting in Γ the rectangle consisting of the modules

$$Z_{ij} = \left(F, X_i \oplus Y_j, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \quad \text{for } i \geqslant 0,\ 1 \leqslant j \leqslant t,$$

and $X_i' = (F, X_i, 1)$ for $i \geqslant 0$. The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geqslant 1$, $j \geqslant 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geqslant 1$, $\tau' Z_{0j} = Y_{j-1}$ if $j \geqslant 2$, Z_{01} is projective, $\tau' X_0' = Y_t$, $\tau' X_i' = Z_{i-1,t}$ if $i \geqslant 1$, $\tau'(\tau^{-1} X_i) = X_i'$ provided X_i is not an injective A-module, otherwise X_i' is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ, or Γ_D, respectively.

If $t = 0$ we define the modified algebra A' to be the one-point extension $A' = A[X]$ and the modified translation quiver Γ' to be the translation quiver obtained from Γ by inserting only the sectional path consisting of the vertices X'_i, $i \geqslant 0$.

The nonnegative integer t is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangle equals $t + 1$. We call t the *parameter* of the operation.

Since Γ is a generalized standard family of components of Γ_A, we then have

Lemma 3.2. *Γ' is a generalized standard family of components of $\Gamma_{A'}$.*

In case Γ is a stable tube, it is clear that any module on the mouth of Γ satisfies the condition for being a pivot for the above operation. Actually, the above operation is, in this case, the tube insertion as considered in [9].

(ad 2) Suppose that $\mathcal{S}(X)$ admits two sectional paths starting at X, one infinite and the other finite with at least one arrow:

$$Y_t \leftarrow \cdots \leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \to X_1 \to X_2 \to \cdots$$

where $t \geqslant 1$. In particular, X is necessarily injective. We define the *modified algebra A'* of A to be the one-point extension $A' = A[X]$ and the *modified translation quiver Γ'* of Γ to be obtained by inserting in Γ the rectangle consisting of the modules

$$Z_{ij} = \left(F, X_i \oplus Y_j, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \quad \text{for } i \geqslant 1,\ 1 \leqslant j \leqslant t,$$

and $X'_i = (F, X_i, 1)$ for $i \geqslant 1$. The translation τ' of Γ' is defined as follows: X'_0 is projective–injective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geqslant 2$, $j \geqslant 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geqslant 1$, $\tau' Z_{1j} = Y_{j-1}$ if $j \geqslant 2$, $\tau' X'_i = Z_{i-1,t}$ if $i \geqslant 2$, $\tau' X'_1 = Y_t$, $\tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not an injective A-module, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ.

The integer $t \geqslant 1$ is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangle equals $t + 1$. We call t the *parameter* of the operation.

Since Γ is a generalized standard family of components of Γ_A, we then have

Lemma 3.3. *Γ' is a generalized standard family of components of $\Gamma_{A'}$.*

(ad 3) Assume $\mathcal{S}(X)$ is the mesh-category of two parallel sectional paths:

$$\begin{tikzcd}
 Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots \arrow[r] & Y_t & & \\
X = X_0 \arrow[r] \arrow[u] & X_1 \arrow[r] \arrow[u] & \cdots \arrow[r] & X_{t-1} \arrow[r] \arrow[u] & X_t \arrow[r] & \cdots
\end{tikzcd}$$

where $t \geqslant 2$. In particular, X_{t-1} is necessarily injective. Moreover, we consider the translation quiver $\overline{\Gamma}$ of Γ obtained by deleting the arrows $Y_i \to \tau_A^{-1} Y_{i-1}$. We assume that the union $\widehat{\Gamma}$ of connected components of $\overline{\Gamma}$ containing the vertices $\tau_A^{-1} Y_{i-1}$, $2 \leqslant i \leqslant t$, is a finite translation quiver. Then $\overline{\Gamma}$ is a disjoint union of $\widehat{\Gamma}$ and a cofinite full translation subquiver Γ^*, containing the pivot X. We define the *modified algebra* A' of A to be the one-point extension $A' = A[X]$ and the *modified translation quiver* Γ' of Γ to be obtained from Γ^* by inserting the rectangle consisting of the modules

$$Z_{ij} = \left(F, X_i \oplus Y_j, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \quad \text{for } i \geqslant 1,\ 1 \leqslant j \leqslant t,$$

and $X'_i = (F, X_i, 1)$ for $i \geqslant 1$. The translation τ' of Γ' is defined as follows: X'_0 is projective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geqslant 2$, $2 \leqslant j \leqslant t$, $\tau' Z_{i1} = X_{i-1}$ if $i \geqslant 1$, $\tau' X'_i = Y_i$ if $1 \leqslant i \leqslant t$, $\tau' X'_i = Z_{i-1,t}$ if $i \geqslant t+1$, $\tau' Y_j = X'_{j-2}$ if $2 \leqslant j \leqslant t$, $\tau'(\tau^{-1} X_i) = X'_i$, if $i \geqslant t$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ^*. We note that X'_{t-1} is injective.

The integer $t \geqslant 2$ is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangle equals $t+1$. We call t the *parameter* of the operation.

Since Γ is a generalized standard family of components of Γ_A, we then have

Lemma 3.4. *Γ' is a generalized standard family of components of $\Gamma_{A'}$.*

(ad 4) Suppose that $\mathcal{S}(X)$ consists an infinite sectional path, starting at X

$$X = X_0 \to X_1 \to X_2 \to \cdots \quad \text{and} \quad Y = Y_1 \to Y_2 \to \cdots \to Y_t$$

with $t \geqslant 1$, be a finite sectional path in Γ_A such that $F_Y = F = F_X$. Let r be a positive integer. Moreover, we consider the translation quiver $\overline{\Gamma}$ of Γ obtained by deleting the arrows $Y_i \to \tau_A^{-1} Y_{i-1}$. We assume that the union $\widehat{\Gamma}$ of connected components of $\overline{\Gamma}$ containing the vertices $\tau_A^{-1} Y_{i-1}$, $2 \leqslant i \leqslant t$, is a finite translation quiver. Then $\overline{\Gamma}$ is a disjoint union of $\widehat{\Gamma}$ and a cofinite full translation subquiver Γ^*, containing the pivot X. For $r = 0$ we define the *modified algebra* A' of A to be the one-point extension $A' = A[X \oplus Y]$ and the *modified translation quiver* Γ' of Γ to be obtained from Γ^* by inserting the rectangle consisting of the modules

$$Z_{ij} = \left(F, X_i \oplus Y_j, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \quad \text{for } i \geqslant 0,\ 1 \leqslant j \leqslant t,$$

and $X'_i = (F, X_i, 1)$ for $i \geqslant 1$. The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geqslant 1$, $j \geqslant 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geqslant 1$, $\tau' Z_{0j} = Y_{j-1}$ if $j \geqslant 2$, Z_{01} is projective, $\tau' X'_0 = Y_t$, $\tau' X'_i = Z_{i-1,t}$ if $i \geqslant 1$, $\tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ^*.

For $r \geqslant 1$, let $G = T_r(F)$, $U_{1,t+1}$, $U_{2,t+1}$, $\dots$, $U_{r,t+1}$ denote the indecomposable projective G-modules, $U_{r,t+1}$, $U_{r,t+2}$, $\dots$, $U_{r,t+r}$ denote the indecomposable injective G-modules, with $U_{r,t+1}$ the unique indecomposable projective–injective G-module. We define the *modified algebra* A' of A to be the triangular matrix algebra of the form:

$$A' = \begin{bmatrix} A & 0 & 0 & \dots & 0 & 0 \\ Y & F & 0 & \dots & 0 & 0 \\ Y & F & F & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Y & F & F & \dots & F & 0 \\ X \oplus Y & F & F & \dots & F & F \end{bmatrix}$$

with $r+2$ columns and rows and the *modified translation quiver* Γ' of Γ to be obtained from Γ^* by inserting the rectangles consisting of the modules $U_{kl} = Y_l \oplus U_{k,t+k}$ for $1 \leqslant k \leqslant r$, $1 \leqslant l \leqslant t$, and

$$Z_{ij} = \left(F, X_i \oplus U_{rj}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \quad \text{for } i \geqslant 0,\ 1 \leqslant j \leqslant t+r,$$

and $X'_i = (F, X_i, 1)$ for $i \geqslant 0$. The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geqslant 1$, $j \geqslant 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geqslant 1$, $\tau' Z_{0j} = U_{r,j-1}$ if $2 \leqslant j \leqslant t+r$, $Z_{01}, U_{k1}, 1 \leqslant k \leqslant r$ are projective, $\tau' U_{kl} = U_{k-1,l-1}$ if $2 \leqslant k \leqslant r$, $2 \leqslant l \leqslant t+r$, $\tau' U_{1l} = Y_{l-1}$ if $2 \leqslant l \leqslant t+1$, $\tau' X'_0 = U_{r,t+r}$, $\tau' X'_i = Z_{i-1,t+r}$ if $i \geqslant 1$, $\tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ^*, or Γ_G, respectively.

We note that the quiver $Q_{A'}$ of A' is obtained from the quiver of the double one-point extension $A[X][Y]$ by adding a path of length $r+1$ with source at the extension vertex of $A[X]$ and sink at the extension vertex of $A[Y]$.

The integers $t \geqslant 1$ and $r \geqslant 0$ are such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangles equals $t+r+1$. We call $t+r$ the *parameter* of the operation.

Since Γ is a generalized standard family of components of Γ_A, we then have

Lemma 3.5. *Γ' is a generalized standard family of components of $\Gamma_{A'}$.*

(fad 1) Assume $\mathcal{S}(X)$ consists of a finite sectional path starting at X:

$$X = X_0 \to X_1 \to X_2 \to \cdots \to X_s$$

where $s \geqslant 0$ and X_s is injective. Let $t \geqslant 1$ be a positive integer, $D = T_t(F)$ and Y_1, Y_2, $\dots$, Y_t denote the indecomposable injective D-modules with $Y = Y_1$ the unique indecomposable projective–injective D-module. We define the *modified algebra* A' of A to be the one-point extension

$$A' = (A \times D)[X \oplus Y]$$

and the *modified translation quiver* Γ' of Γ to be obtained by inserting in Γ the rectangle consisting of the modules

$$Z_{ij} = \left(F, X_i \oplus Y_j, \begin{bmatrix}1\\1\end{bmatrix}\right) \quad \text{for } 0 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant t,$$

$X'_i = (F, X_i, 1)$ for $0 \leqslant i \leqslant s$, $Y'_j = (F, Y_j, 1)$ for $1 \leqslant j \leqslant t$, and $W = S_p$, where p is the extension vertex of $A[X]$. The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geqslant 1$, $j \geqslant 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geqslant 1$, $\tau' Z_{0j} = Y_{j-1}$ if $j \geqslant 2$, Z_{01} is projective, $\tau' X'_0 = Y_t$, $\tau' X'_i = Z_{i-1,t}$ if $i \geqslant 1$, $\tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ', $\tau' Y'_1 = X_s$, $\tau' Y'_j = Z_{s,j-1}$ if $2 \leqslant j \leqslant t$, $\tau' W = Z_{st}$. For the remaining vertices of Γ', τ' coincides with the translation of Γ, or Γ_D, respectively. If $t = 0$ we define the modified algebra A' to be the one-point extension $A' = A[X]$ and the modified translation quiver Γ' to be the component obtained from Γ by inserting only the sectional path consisting of the vertices X'_i, $0 \leqslant i \leqslant s$, and W.

Observe that for $s = 0 = t$ the new translation quiver Γ' is obtained from Γ by adding the projective–injective vertex X'_0 and the vertex W with $\tau' W = X_0$.

(fad 2) Suppose that $\mathcal{S}(X)$ admits two finite sectional paths starting at X, each of them with at least one arrow:

$$Y_t \leftarrow \cdots \leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_s$$

where $t \geqslant 1$ and $s \geqslant 1$. In particular, X and X_s are necessarily injective. We define the *modified algebra* A' of A to be the one-point extension $A' = A[X]$ and the *modified translation quiver* Γ' of Γ to be obtained by inserting in Γ the rectangle consisting of the modules

$$Z_{ij} = \left(F, X_i \oplus Y_j, \begin{bmatrix}1\\1\end{bmatrix}\right) \quad \text{for } 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant t,$$

$X'_i = (F, X_i, 1)$ for $1 \leqslant i \leqslant s$, $Y'_j = (F, Y_j, 1)$ for $1 \leqslant j \leqslant t$, and $W = S_p$, where p is the extension vertex of $A[X]$. The translation τ' of Γ' is defined as follows: X'_0 is projective–injective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geqslant 2$, $j \geqslant 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geqslant 1$, $\tau' Z_{1j} = Y_{j-1}$ if $j \geqslant 2$, $\tau' X'_i = Z_{i-1,t}$ if $i \geqslant 2$, $\tau' X'_1 = Y_t$, $\tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ', $\tau' Y'_1 = X_s$, $\tau' Y'_j = Z_{s,j-1}$ if $2 \leqslant j \leqslant t$, $\tau' W = Z_{st}$. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ.

(fad 3) Assume $\mathcal{S}(X)$ is the mesh-category of two finite parallel sectional paths:

$$\begin{tikzcd}
 & Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots \arrow[r] & Y_t & & & \\
X = X_0 \arrow[r] \arrow[ru, phantom, ""] & X_1 \arrow[r] \arrow[u] & \cdots \arrow[r] & X_{t-1} \arrow[r] & X_t \arrow[r] & \cdots \arrow[r] & X_s
\end{tikzcd}$$

where $s \geqslant t-1$, $t \geqslant 2$. In particular, X_{t-1} and X_s are necessarily injective. We define the *modified algebra* A' of A to be the one-point extension $A' = A[X]$ and the *modified translation quiver* Γ' of Γ to be obtained by inserting in Γ the rectangle consisting of the modules

$$Z_{ij} = \left(F, X_i \oplus Y_j, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \quad \text{for } 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant t,$$

$X'_i = (F, X_i, 1)$ for $1 \leqslant i \leqslant s$, $Y'_j = (F, Y_j, 1)$ for $1 \leqslant j \leqslant t$, and $W = S_p$, where p is the extension vertex of $A[X]$. The translation τ' of Γ' is defined as follows: X'_0 is projective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geqslant 2$, $2 \leqslant j \leqslant t$, $\tau' Z_{i1} = X_{i-1}$ if $i \geqslant 1$, $\tau' X'_i = Y_i$ if $1 \leqslant i \leqslant t$, $\tau' X'_i = Z_{i-1,t}$ if $i \geqslant t+1$, $\tau' Y_j = X'_{j-2}$ if $2 \leqslant j \leqslant t$, $\tau'(\tau^{-1} X_i) = X'_i$, if $i \geqslant t$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ'. In both cases, X'_{t-1} is injective, $\tau' Y'_1 = X_s$, $\tau' Y'_j = Z_{s,j-1}$ if $2 \leqslant j \leqslant t$, $\tau' W = Z_{st}$. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ. Observe that for $s = t-1$ we have $Z_{tt} = Y'_t$ and $X'_t = W$.

(fad 4) Suppose that $\mathcal{S}(X)$ consists of a finite sectional path starting at X:

$$X = X_0 \to X_1 \to X_2 \to \cdots \to X_s$$

with $s \geqslant 1$ and X_s injective, and

$$Y = Y_1 \to Y_2 \to \cdots \to Y_t$$

$t \geqslant 1$, be a finite sectional path in Γ_A such that $F_Y = F = F_X$. Let r be a positive integer. For $r = 0$ we define the *modified algebra* A' of A to be the one-point extension $A' = A[X \oplus Y]$ and the *modified translation quiver* Γ' of Γ to be obtained by inserting in Γ the rectangle consisting of the modules

$$Z_{ij} = \left(F, X_i \oplus Y_j, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \quad \text{for } 0 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant t,$$

$X'_i = (F, X_i, 1)$ for $0 \leqslant i \leqslant s$, $Y'_j = (F, Y_j, 1)$ for $1 \leqslant j \leqslant t$, and $W = S_p$, where p is the extension vertex of $A[X]$. The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geqslant 1$, $j \geqslant 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geqslant 1$, $\tau' Z_{0j} = Y_{j-1}$ if $j \geqslant 2$, Z_{01} is projective, $\tau' X'_0 = Y_t$, $\tau' X'_i = Z_{i-1,t}$ if $i \geqslant 1$, $\tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ', $\tau' Y'_1 = X_s$, $\tau' Y'_j = Z_{s,j-1}$ if $2 \leqslant j \leqslant t$, $\tau' W = Z_{st}$. For the remaining vertices of Γ', τ' coincides with the translation of Γ.

For $r \geqslant 1$, let $G = T_r(F)$, $U_{1,t+1}$, $U_{2,t+1}$, $\ldots$, $U_{r,t+1}$ denote the indecomposable projective G-modules, $U_{r,t+1}$, $U_{r,t+2}$, $\ldots$, $U_{r,t+r}$ denote the indecomposable injective G-modules, with $U_{r,t+1}$ the unique indecomposable projective–injective G-module. We define the *modified algebra* A' of A to be the triangular matrix algebra of the form:

$$A' = \begin{bmatrix} A & 0 & 0 & \dots & 0 & 0 \\ Y & F & 0 & \dots & 0 & 0 \\ Y & F & F & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Y & F & F & \dots & F & 0 \\ X \oplus Y & F & F & \dots & F & F \end{bmatrix}$$

with $r+2$ columns and rows and the *modified translation quiver* Γ' of Γ to be obtained by inserting in Γ the rectangles consisting of the modules $U_{kl} = Y_l \oplus U_{k,t+k}$ for $1 \leqslant k \leqslant r$, $1 \leqslant l \leqslant t$,

$$Z_{ij} = \left(F, X_i \oplus U_{rj}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \quad \text{for } 0 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant t+r,$$

$X'_i = (F, X_i, 1)$ for $0 \leqslant i \leqslant s$, $Y'_j = (F, U_{rj}, 1)$ for $1 \leqslant j \leqslant t+r$, and $W = S_p$, where p is the extension vertex of $A[X]$. The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geqslant 1$, $j \geqslant 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geqslant 1$,$\tau' Z_{0j} = U_{r,j-1}$ if $2 \leqslant j \leqslant t+r$, Z_{01}, U_{k1}, $1 \leqslant k \leqslant r$ are projective, $\tau' U_{kl} = U_{k-1,l-1}$ if $2 \leqslant k \leqslant r$, $2 \leqslant l \leqslant t+r$, $\tau' U_{1l} = Y_{l-1}$ if $2 \leqslant l \leqslant t+1$, $\tau' X'_0 = U_{r,t+r}$, $\tau' X'_i = Z_{i-1,t+r}$ if $i \geqslant 1$, $\tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ', $\tau' Y'_1 = X_s$, $\tau' Y'_j = Z_{s,j-1}$ if $2 \leqslant j \leqslant t+r$, $\tau' W = Z_{s,t+r}$. For the remaining vertices of Γ', τ' coincides with the translation of Γ, or Γ_G, respectively.

(ad 5) We define the *modified algebra* A' of A to be the iteration of the extensions described in the definitions of the admissible operations (ad 1)–(ad 4), and their finite versions corresponding to the operations (fad 1)–(fad 4). The *modified translation quiver* Γ' of Γ is obtained in the following three steps: first we are doing on Γ one of the operations (fad 1), (fad 2) or (fad 3), next a finite number (possibly empty) of the operation (fad 4) and finally the operation (ad 4), and in such a way that the sectional paths starting from all the new projective vertices have a common cofinite (infinite) sectional subpath.

Since Γ is a generalized standard family of components of Γ_A, we then have

Lemma 3.6. *Γ' is a generalized standard family of components of $\Gamma_{A'}$.*

Finally, together with each of the admissible operations (ad 1)–(ad 5), we consider its dual, denoted by (ad 1*)–(ad 5*). These ten operations are called the *admissible operations*. Following [25] a connected translation quiver Γ is said to be a *generalized multicoil* if Γ can be obtained from a finite family $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_s$ of stable tubes by an iterated application of admissible operations (ad 1), (ad 1*), (ad 2), (ad 2*), (ad 3), (ad 3*), (ad 4), (ad 4*), (ad 5) or (ad 5*). If $s = 1$, such a translation quiver Γ is said to be a *generalized coil*. The admissible operations of types (ad 1), (ad 2), (ad 3), (ad 1*), (ad 2*) and (ad 3*) have been introduced in [2–4], and the admissible operations (ad 4) and (ad 4*) for $r = 0$ in [24]. We refer also to [26] for the structure of indecomposable modules lying in (generalized) standard coils.

Observe that any stable tube is trivially a generalized coil. A *tube* (in the sense of [9]) is a generalized coil having the property that each admissible operation in the sequence defining it is of the form (ad 1) or (ad 1*). If we apply only operations of type (ad 1) (respectively of type (ad 1*)) then such a generalized coil is called a *ray tube* (respectively a *coray tube*). Observe that a generalized coil without injective (respectively projective) vertices is a ray tube (respectively a coray tube). A *quasi-tube* (in the sense of [36]) is a generalized coil having the property that each of the admissible operations in the sequence defining it is of type (ad 1), (ad 1*), (ad 2) or (ad 2*). Finally, following [3] a coil is a generalized coil having the property that each of the admissible operations in the sequence defining it is one of the forms (ad 1), (ad 1*), (ad 2), (ad 2*), (ad 3) or (ad 3*). We note that any generalized multicoil Γ is a coherent translation quiver with trivial valuations and its cyclic part ${}_c\Gamma$ (the translation subquiver of Γ obtained by removing from Γ all acyclic vertices and the arrows attached to them) is infinite, connected and cofinite in Γ, and so Γ is almost cyclic.

Finally, let C be a (not necessarily connected) concealed canonical algebra and $\mathcal{T}_C$ a separating family of stable tubes of Γ_C. We say that an algebra is a *generalized multicoil enlargement* of C using modules from $\mathcal{T}_C$ if A is obtained from C by an iteration of admissible operations of types (ad 1)–(ad 5) and (ad 1*)–(ad 5*) performed either on stable tubes of $\mathcal{T}_C$, or on generalized multicoils obtained from stable tubes of $\mathcal{T}_C$ by means of operations done so far. We note that a generalized multicoil enlargement A of C invoking only admissible operations of type (ad 1) (respectively of type (ad 1*)) is a tubular extension (respectively tubular coextension) of C in the sense of [34].

The following proposition is a direct consequence of Proposition 3.1, Lemmas 3.2–3.6, and the corresponding definitions.

Proposition 3.7. *Let C be a concealed canonical algebra, $\mathcal{T}_C$ a separating family of stable tubes of Γ_C, and A a generalized multicoil enlargement of C using modules from $\mathcal{T}_C$. Then Γ_A admits a generalized standard family $\mathcal{C}_A$ of generalized multicoils obtained from the family $\mathcal{T}_C$ of stable tubes by a sequence of admissible operations corresponding to the admissible operations leading from C to A.*

We will need also the following technical lemmas.

Lemma 3.8. *Let A be an algebra, Γ be a generalized standard component of Γ_A and $X \in \Gamma$ be an* (ad 4) *or* (ad 4*)*-pivot. Let A' be the modified algebra and Γ' be the modified component. Then any indecomposable A'-module whose restriction to A has an indecomposable direct summand of the form X_i or Y_j, for some $i \geqslant 0$, $1 \leqslant j \leqslant t$, belongs to Γ'.*

Proof. Similar to the proof of [4, (2.4)]. □

Lemma 3.9. *Let A be an algebra with a family $\mathcal{C}_A$ of generalized multicoils separating $\mathcal{P}_A$ from $\mathcal{Q}_A$, Γ be a generalized multicoil in $\mathcal{C}_A$ and X be an* (ad 4)*-pivot in Γ and*

$$A' = \begin{bmatrix} A & 0 & 0 & \dots & 0 & 0 \\ Y & F & 0 & \dots & 0 & 0 \\ Y & F & F & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Y & F & F & \dots & F & 0 \\ X \oplus Y & F & F & \dots & F & F \end{bmatrix}$$

be the modified algebra. Let e denote the extension vertex of $A[X]$, and $\mathcal{P}'_{A'}$, $\mathcal{C}'_{A'}$, $\mathcal{Q}'_{A'}$ be the classes in ind A' *defined as follows*:

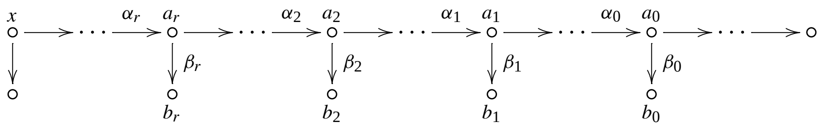
(i) $\mathcal{P}'_{A'} = \mathcal{P}_A$.
(ii) $\mathcal{C}'_{A'}$ *consists of all indecomposables $M_{A'}$ such that $M_e = 0$ and $M = M|_A$ is in $(\mathcal{C}_A \setminus \Gamma) \cup \Gamma^*$* (*where Γ^*—as in the definition of* (ad 4)), *or $M_e = 0$ and $M_f \neq 0$ where $f \neq e$ denotes one of the r added vertices to Q_A, or $M_e \neq 0$ and $M|_A$ has an indecomposable direct summand of the form X_i, for some $i \geqslant 0$.*
(iii) $\mathcal{Q}'_{A'}$ *consists of all indecomposables $M_{A'}$ such that $M_e = 0$ and $M = M|_A$ is in $\mathcal{Q}_A \cup (\Gamma \setminus \Gamma^*)$, or $M = (F, 0, 0)$, or $M_e \neq 0$ and the indecomposable direct summands of $M|_A$ belong either to the set $\{Y_1, Y_2, \dots, Y_t\}$ or to the support of* $\operatorname{Hom}_A(X, -)|_{\mathcal{Q}_A}$.

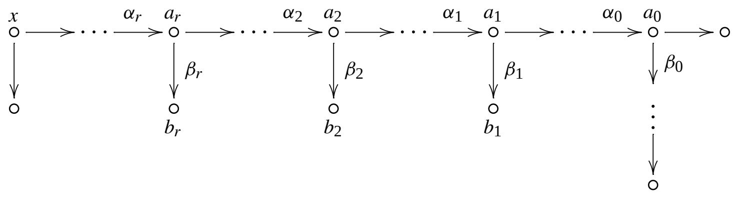
Then ind $A' = \mathcal{P}'_{A'} \vee \mathcal{C}'_{A'} \vee \mathcal{Q}'_{A'}$, *and $\mathcal{C}'_{A'}$ separates $\mathcal{P}'_{A'}$ from $\mathcal{Q}'_{A'}$.*

Proof. Similar to the proof of [4, (2.5) and (2.6)] involving additionally Lemmas 3.5 and 3.8. $\square$

Lemma 3.10. *Let A be a multicoil enlargement of a concealed canonical algebra C, and assume that the admissible operation* (ad 5) *is applied. Then this admissible operation* (ad 5) *can be replaced by an admissible operation* (ad 1) *followed by a finite number of admissible operations of type* (ad 5*).

Proof. The part of the bound quiver Q_A obtained by applying the admissible operation (ad 5) can be visualized as follows:





$$
\begin{tikzcd}
\overset{x}{\circ} \arrow[r] \arrow[d] & \cdots \arrow[r, "\alpha_r"] & \overset{a_r}{\circ} \arrow[r] \arrow[d, "\beta_r"] & \cdots \arrow[r, "\alpha_2"] & \overset{a_2}{\circ} \arrow[r] \arrow[d, "\beta_2"] & \cdots \arrow[r, "\alpha_1"] & \overset{a_1}{\circ} \arrow[r] \arrow[d, "\beta_1"] & \cdots \arrow[r, "\alpha_0"] & \overset{a_0}{\circ} \arrow[r] \arrow[d, "\beta_0"] & \circ \\
\circ & & \underset{b_r}{\circ} & & \underset{b_2}{\circ} & & \underset{b_1}{\circ} & & \underset{b_0}{\circ} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\overset{x}{\circ} \arrow[r] \arrow[d] & \cdots \arrow[r, "\alpha_r"] & \overset{a_r}{\circ} \arrow[r] \arrow[d, "\beta_r"] & \cdots \arrow[r, "\alpha_2"] & \overset{a_2}{\circ} \arrow[r] \arrow[d, "\beta_2"] & \cdots \arrow[r, "\alpha_1"] & \overset{a_1}{\circ} \arrow[r] \arrow[d, "\beta_1"] & \cdots \arrow[r, "\alpha_0"] & \overset{a_0}{\circ} \arrow[r] & \circ \\
\circ & & \underset{b_r}{\circ} & & \underset{b_2}{\circ} & & \underset{b_1}{\circ} & & &
\end{tikzcd}
$$

where $\alpha_i\beta_i = 0$ for $0 \leqslant i \leqslant r$, a_0 is the extension point of (fad 1), (fad 2) or (fad 3), $a_1, a_2, \ldots, a_r$ are the extension points of r consecutive operations of type (fad 4) and x is the extension point of the admissible operation (ad 4), which finishes the whole process of creating (ad 5). The first two figures show the cases when in the admissible operation (ad 5) we start with (fad 1), the third figure shows the cases when in the admissible operation (ad 5) we start with (fad 2), and the third and the fourth figures show the cases when we start with (fad 3). It is easily seen that x is also the extension point of the admissible operation (ad 1) which contains the horizontal arrows. Note that these arrows can define whole operation (ad 1) or its part. Moreover, $b_1, b_2, \ldots, b_r$ are the coextension points of r consecutive operations of type (fad 1*), b_0 is also the coextension point of the operation (fad 1*) except the case when in the starting operation (fad 3) we have $s = t - 1$. In this case b_0 is the coextension point of the operation (fad 3*). So, in the cases visualized in the first and the third figures we have $r + 1$ consecutive operations of type (ad 5*), and in the other cases we have r consecutive operations of type (ad 5*). Observe that the above admissible operations (ad 5*) are of the form (fad 1*) and (ad 4*), or (fad 3*) and (ad 4*). $\square$

4. Proofs of Theorems A and C

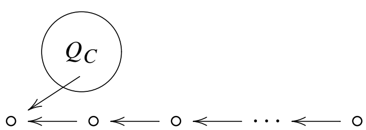
Let K be a truncated branch at a vertex x, and $A = kQ/I$ be a bound quiver algebra over a field k, where $Q = Q_A$ is the ordinary quiver of A and I is an admissible ideal of the path algebra kQ of Q, and $E \in \operatorname{mod} A$. Recall that the *truncated branch extension* $A[E, K]$ by the truncated branch K [34, (4.7)] is constructed in the following way: to the one-point extension $A[E]$ with extension vertex w (that is, $\operatorname{rad} P_w = E$) we add the truncated branch K by identifying the vertices x and w. If $E_1, E_2, \ldots, E_n \in \operatorname{mod} A$ and $K_1, K_2, \ldots, K_n$ is a set of truncated branches, then the truncated branch extension $A[E_i, K_i]_{i=1}^n$ is defined inductively as $A[E_i, K_i]_{i=1}^n = (A[E_i, K_i]_{i=1}^{n-1})[E_n, K_n]$. The concept of *truncated branch coextension* is defined dually.

Now, we will present a simultaneous proof of Theorems A and C. First we note that if A is an algebra with a separating family $\mathcal{C}_A$ of almost cyclic coherent components of Γ_A then clearly $\mathcal{C}_A$ is a sincere generalized standard family of almost cyclic coherent components without external short paths, and hence the implication (i) $\Rightarrow$ (ii) of Theorem A holds.

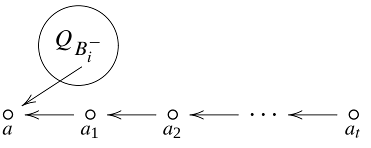
Assume now that A is an algebra with a sincere generalized standard family $\mathcal{C}_A$ of almost cyclic coherent components in Γ_A without external short paths. We claim that then A is a generalized multicoil enlargement of a concealed canonical algebra C, and hence the implication (ii) $\Rightarrow$ (iii) of Theorem A holds. It follows from [25, Theorem A] that the family $\mathcal{C}_A$ of generalized multicoils can be obtained, as the translation quiver, from a family $\mathcal{T}$ of stable tubes by a sequence of admissible operations of types (ad 1)–(ad 5) and their duals. Denote by C the support algebra of the family $\mathcal{T}$ of indecomposable A-modules. Since $\mathcal{C}_A$ is a sincere generalized standard family of Γ_A, invoking results established in Section 3, we conclude that $\mathcal{T}$ is a sincere generalized standard family of stable tubes of Γ_C and A is a generalized standard multicoil enlargement of C using modules from $\mathcal{T}$ and admissible operations of types (ad 1)–(ad 5) and their duals, corresponding to the admissible operations leading from $\mathcal{T}$ to $\mathcal{C}_A$. Moreover, $\mathcal{T}$ is a sincere family of pairwise orthogonal stable tubes of Γ_C without external short paths. Applying Theorem 2.1 we conclude that C is a (not necessarily connected) concealed canonical algebra. Therefore, A is a generalized multicoil enlargement of the concealed canonical algebra C, and this proves our claim.

Our next aim is to prove Theorem C. From the above considerations, we may assume that A is a generalized multicoil enlargement of a concealed canonical algebra C. Let $\mathcal{T}_C$ be a separating family of stable tubes of Γ_C, and $\operatorname{ind} C = \mathcal{P}_C \vee \mathcal{T}_C \vee \mathcal{Q}_C$ the induced decomposition of $\operatorname{ind} C$. Let $C = C_1 \times C_2 \times \cdots \times C_m$ be decomposition of C into product of connected algebras.

Let A_l be a unique maximal convex truncated branch coextension of C inside A, that is, $A_l = B_1^- \times B_2^- \times \cdots \times B_m^-$, where B_i^- is a unique maximal convex truncated branch coextension of C_i inside A, $1 \leqslant i \leqslant m$. We shall prove that A_l is the required quasitilted algebra and $\mathcal{P}_A = \mathcal{P}_{B_1^-} \vee \mathcal{P}_{B_2^-} \vee \cdots \vee \mathcal{P}_{B_m^-}$ by induction on the number n of admissible operations leading from C to the algebra A. If $n = 1$, then we can only apply an admissible operation (ad 1) or (ad 1*), so $m = 1$. If the algebra A is obtained from C by applying (ad 1), then $A_l = C$ and there is nothing to show. If the algebra A is obtained from C by applying (ad 1*), then it is clear that $A_l = A$ is a unique maximal truncated branch coextension of C and $\mathcal{P}_A = \mathcal{P}_{A_l} = \mathcal{P}_{B_1^-}$. Moreover, the bound quiver of A_l is of the form



Let $n > 1$. Assume that the statement holds for $n-1$, so after applying $n-1$ admissible operations we have $A_l = B_1^- \times B_2^- \times \cdots \times B_m^-$, where B_i^- is a unique maximal convex truncated branch coextension of C_i (that is, $B_i^- = {}_{j=1}^{t_i}[K_j, E_j]C_i$, where $K_1, K_2, \ldots, K_{t_i}$ are truncated branches), $1 \leqslant i \leqslant m$, and $\mathcal{P}_A = \mathcal{P}_{B_1^-} \vee \mathcal{P}_{B_2^-} \vee \cdots \vee \mathcal{P}_{B_m^-}$. Note that we can apply an admissible operation (ad 2), (ad 3) or (ad 4) (that is also (ad 5)) (respectively (ad 2*), (ad 3*), (ad 4*)) if the number of all successors (respectively all predecessors) of the module Y_i (which occurs in the definitions of the above admissible operations) is finite for each $1 \leqslant i \leqslant t$. If it is not the case, then the family of generalized multicoils obtained after applying such admissible operation is not sincere, and then is not separating. If the nth admissible operation is of one of types (ad 1)–(ad 5), then A_l does not change, therefore the proof follows from inductive assumption. If it is of type (ad 1*) then let A' be the algebra obtained from A by applying this admissible operation with pivot X, and $a_1, a_2, \dots, a_t$ be the points in the quiver $Q_{A'}$ of A' corresponding to the new indecomposable injective A'-modules. Then $A'_l = B_1^- \times B_2^- \times \cdots \times B_{i-1}^- \times B_i'^- \times B_{i+1}^- \times \cdots \times B_m^-$, where the bound quiver of $B_i'^-$ is of the form



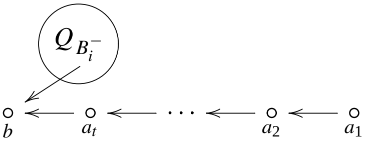
for some $1 \leqslant i \leqslant m$. Then $B_i'^-$ is the coextension of B_i^- at X by the coextension truncated branch K consisting of the points $a, a_1, a_2, \dots, a_t$, that is, we have $B_i'^- = [K, X]B_i^-$. Therefore A'_l has the required form. From the dual fact to [34, (4.7)(1)] we know that $\mathcal{P}_{A'}$ is the module class given by all indecomposable A'-modules M with either $M|_A \neq 0$ and $M \in \mathcal{P}_A$, or else the support of M is contained in branch K and $\langle \underline{\dim} M, l_K \rangle > 0$, where l_K is the branch length function (see [34, (4.4)]). Therefore,

$$\mathcal{P}_{A'} = \mathcal{P}_{B_1^-} \vee \mathcal{P}_{B_2^-} \vee \cdots \vee \mathcal{P}_{B_{i-1}^-} \vee \mathcal{P}_{B_i'^-} \vee \mathcal{P}_{B_{i+1}^-} \vee \cdots \vee \mathcal{P}_{B_m^-},$$

where

$$A'_l = B_1^- \times B_2^- \times \cdots \times B_{i-1}^- \times B_i'^- \times B_{i+1}^- \times \cdots \times B_m^-,$$

$B_i'^- = [K, X]B_i^-$ and $B_i'^-, B_1^-, B_2^-, \dots, B_{i-1}^-, B_{i+1}^-, \dots, B_m^-$ are the unique maximal convex truncated branch coextensions of C_i inside A, $1 \leqslant i \leqslant m$. If the nth admissible operation is of type (ad 2*), then in the sequence of earlier $n-1$ admissible operations, there is an operation of type (ad 1) or (ad 5) which contains an operation (fad 1) which gives rise to the pivot X of (ad 2*), and the operations done between these two must not affect the support of $\mathrm{Hom}_A(-, X)$ restricted to the generalized multicoil containing X. Note that in general, in the sequence of earlier $n-1$ admissible operations can be an operation of type (ad 5*) which contains an operation (fad 4*) which gives rise to the pivot X of (ad 2*) but from the dual fact to Lemma 3.10 this case can be reduced to (ad 5) which contains an operation (fad 1). Let A' be the algebra obtained from A by applying (ad 2*) with pivot X, and $a, a_1, a_2, \dots, a_t$ ($X = P_a$) be the points in the quiver $Q_{A'}$ of A' corresponding to the new indecomposable projective A'-modules obtained after performing the above admissible operation (ad 1) or the operation (fad 1). Then $A'_l = B_1^- \times B_2^- \times \cdots \times B_{i-1}^- \times B_i'^- \times B_{i+1}^- \times \cdots \times B_m^-$, and the bound quiver of $B_i'^-$ is of the form



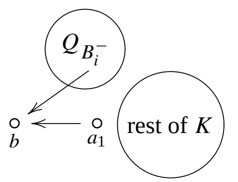
for some $1 \leqslant i \leqslant m$, where b is the coextension point of $A' = [X]A$. Then $B_i'^-$ is the coextension of B_i^- at X by the coextension truncated branch K consisting of the points $b, a_t, a_{t-1}, \ldots, a_1$, that is, we have $B_i'^- = [K, X]B_i^-$. Therefore A_l' has the required form. From the dual fact to [4, (2.5)] we know that $\mathcal{P}_{A'}$ consists of all indecomposable A'-modules M such that $M_b = 0$ and $M = M|_A$ is in $\mathcal{P}_A$, or $M = (0, 0, F)$, or $M_b \neq 0$ and the indecomposable direct summands of $M|_A$ belong either to the set $\{Y_1, Y_2, \ldots, Y_t\}$ or to the support of $\operatorname{Hom}_A(-, X)|_{\mathcal{P}_A}$. Therefore,

$$\mathcal{P}_{A'} = \mathcal{P}_{B_1^-} \vee \mathcal{P}_{B_2^-} \vee \cdots \vee \mathcal{P}_{B_{i-1}^-} \vee \mathcal{P}_{B_i'^-} \vee \mathcal{P}_{B_{i+1}^-} \vee \cdots \vee \mathcal{P}_{B_m^-},$$

where

$$A_l' = B_1^- \times B_2^- \times \cdots \times B_{i-1}^- \times B_i'^- \times B_{i+1}^- \times \cdots \times B_m^-,$$

$B_i'^- = [K, X]B_i^-$ and $B_i'^-, B_1^-, B_2^-, \ldots, B_{i-1}^-, B_{i+1}^-, \ldots, B_m^-$ are the unique maximal convex truncated branch coextensions of C_i inside A, $1 \leqslant i \leqslant m$. If the nth admissible operation is of type (ad 3*), then in the sequence of earlier $n - 1$ admissible operations, there must be at least one operation of type (ad 1) or (ad 5) which contains the operation (fad 1) which gives rise to the pivot X of (ad 3*) and to the modules $Y_1, Y_2, \ldots, Y_t$ in the support of $\operatorname{Hom}_A(-, X)$ restricted to the generalized multicoil containing X. The operations done after must not affect this support. Again, in general, in the sequence of earlier $n - 1$ admissible operations can be an operation of type (ad 5*) which contains an operation (fad 4*) which gives rise to the pivot X of (ad 3*) but from the dual fact to Lemma 3.10 this case can be reduced to (ad 5) which contains an operation (fad 1). Suppose that we had r such consecutive admissible operations of types (ad 1) or (fad 1), the first of which had X_t as a pivot, and these admissible operations built up a branch K in A with points $a, a_1, a_2, \ldots, a_t$ in Q_A, so that X_{t-1} and Y_t are the indecomposable projective A-modules corresponding respectively to a and a_1, and both Y_1 and $\tau_A^{-1} Y_1$ are ray modules in the generalized multicoil containing the (ad 3*)-pivot X. Let A' be the algebra obtained from A by applying (ad 3*) with pivot X. Then $A_l' = B_1^- \times B_2^- \times \cdots \times B_{i-1}^- \times B_i'^- \times B_{i+1}^- \times \cdots \times B_m^-$, and the bound quiver of $B_i'^-$ is of the form



for some $1 \leqslant i \leqslant m$, where b is the coextension point of $A' = [X]A$. Then $B_i'^-$ is the coextension of B_i^- at X by the coextension truncated branch K consisting of the points $b, a_1, a_2, \ldots, a_t$, that is, we have $B_i'^- = [K, X_t]B_i^-$. Therefore A_l' has the required form. From the dual fact to [4, (2.6)] we know that $\mathcal{P}_{A'}$ consists of all indecomposable A'-modules M such that $M_b = 0$ and $M = M|_A$ is in $\mathcal{P}_A \cup (\Gamma \setminus \Gamma^*)$, or $M = (0, 0, F)$, or $M_b \neq 0$ and the indecomposable direct summands of $M|_A$ belong either to the set

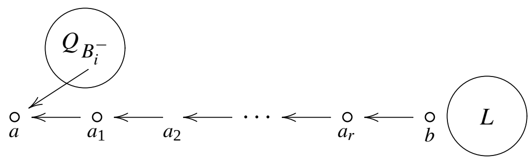
$\{Y_1, Y_2, \ldots, Y_t\}$ or to the support of $\mathrm{Hom}_A(-, X)|_{\mathcal{P}_A}$, where Γ' and Γ^* are as in the definition of (ad 3*). Therefore,

$$\mathcal{P}_{A'} = \mathcal{P}_{B_1^-} \vee \mathcal{P}_{B_2^-} \vee \cdots \vee \mathcal{P}_{B_{i-1}^-} \vee \mathcal{P}_{B_i'^-} \vee \mathcal{P}_{B_{i+1}^-} \vee \cdots \vee \mathcal{P}_{B_m^-},$$

where

$$A_l' = B_1^- \times B_2^- \times \cdots \times B_{i-1}^- \times B_i'^- \times B_{i+1}^- \times \cdots \times B_m^-,$$

$B_i'^- = [K, X_t]B_i^-$ and $B_i'^-, B_1^-, B_2^-, \ldots, B_{i-1}^-, B_{i+1}^-, \ldots, B_m^-$ are the unique maximal convex truncated branch coextensions of C_i inside A, $1 \leqslant i \leqslant m$. If the nth admissible operation is of type (ad 4*), then let A' be the algebra obtained from A by applying this admissible operation with pivot X and the end Y_1 of a finite sectional path $Y_1 \leftarrow Y_2 \leftarrow \cdots \leftarrow Y_t$. Note that this finite sectional path is the linearly oriented quiver of type $\mathbb{A}_t$ and its support algebra Λ (given by the vertices corresponding to the simple composition factors of the modules $Y_1, Y_2, \ldots, Y_t$) is a tilted algebra of the path algebra D of the linearly oriented quiver of type $\mathbb{A}_t$. From [34, (4.4)(2)] we know that Λ is a bound quiver algebra given by a truncated branch in x, where x corresponds to the unique projective–injective D-module. Then $A_l' = B_1^- \times B_2^- \times \cdots \times B_{i-1}^- \times B_i'^- \times B_{i+1}^- \times \cdots \times B_m^-$, and the bound quiver of $B_i'^-$ is of the form



for some $1 \leqslant i \leqslant m$, where the index r is as in the definition of (ad 4*), a is the coextension point of $[X]A$, b is the coextension point of $[Y_1]A'$ and L is a truncated branch in b generated by the support of $Y_1 \oplus Y_2 \oplus \cdots \oplus Y_t$. Then $B_i'^-$ is the coextension of B_i^- by the coextension truncated branch K in a consisting of the points $a, a_1, a_2, \ldots, a_r, b$ and the branch L, that is, we have $B_i'^- = [K, X]B_i^-$. Therefore A_l' has the required form. From the dual fact to Lemma 3.9 we conclude that

$$\mathcal{P}_{A'} = \mathcal{P}_{B_1^-} \vee \mathcal{P}_{B_2^-} \vee \cdots \vee \mathcal{P}_{B_{i-1}^-} \vee \mathcal{P}_{B_i'^-} \vee \mathcal{P}_{B_{i+1}^-} \vee \cdots \vee \mathcal{P}_{B_m^-},$$

where

$$A_l' = B_1^- \times B_2^- \times \cdots \times B_{i-1}^- \times B_i'^- \times B_{i+1}^- \times \cdots \times B_m^-,$$

$B_i'^- = [K, X_t]B_i^-$ and $B_i'^-, B_1^-, B_2^-, \ldots, B_{i-1}^-, B_{i+1}^-, \ldots, B_m^-$ are the unique maximal convex truncated branch coextensions of C_i inside A, $1 \leqslant i \leqslant m$. There remains to consider the case where the nth admissible operation is of type (ad 5*). Since in the definition of admissible operation (ad 5*) we use the finite versions (fad 1*)–(fad 4*) of the admissible operations (ad 1*)–(ad 4*) and the admissible operation (ad 4*), we conclude that this case follows from the above considerations. This finishes the proof of part (i) of Theorem C. The part (ii) of Theorem C follows by dual arguments.

We will show now that the implication (iii) $\Rightarrow$ (i) of Theorem A also holds. Let A be a generalized multicoil enlargement of a concealed canonical algebra C. In the proof of Theorem C we proved that there exist quasitilted factor algebras A_l and A_r of A, satisfying the conditions (i) and (ii), whose quivers Q_{A_l} and Q_{A_r} are full convex subquivers of the quiver Q_A of A. In particular, the modules in $\operatorname{ind} A$ split into three disjoint classes $\mathcal{P}_A = \mathcal{P}_{A_l}$, $\mathcal{C}_A$ and $\mathcal{Q}_A = \mathcal{Q}_{A_r}$ where $\mathcal{C}_A$ is a sincere generalized standard family of generalized multicoils (which are almost cyclic and coherent), $\mathcal{P}_A$ consists of all proper predecessors of $\mathcal{C}_A$ in $\operatorname{ind} A$, and $\mathcal{Q}_A$ consists of all proper successors of $\mathcal{C}_A$ in $\operatorname{ind} A$. Hence the conditions (S1) and (S2) of a separating family are for $\mathcal{P}_A$, $\mathcal{C}_A$ and $\mathcal{Q}_A$ satisfied. In order to prove the condition (S3), take a nonzero morphism $f : X \to Y$ for X in $\mathcal{P}_A$ and Y in $\mathcal{Q}_A$. Since $\mathcal{P}_A = \mathcal{P}_{A_l}$, X is an A_l-module and f is given by the restriction $f|_{A_l} : X|_{A_l} \to Y|_{A_l}$ to A_l. Moreover, we have $\operatorname{ind} A_l = \mathcal{P}_{A_l} \vee \mathcal{T}_{A_l} \vee \mathcal{Q}_{A_l}$, and $Y|_{A_l}$ belongs to $\operatorname{add}(\mathcal{T}_{A_l} \vee \mathcal{Q}_{A_l})$. Let $Y|_{A_l} = Y' \oplus Y''$, where Y' belongs to $\operatorname{add} \mathcal{T}_{A_l}$ and Y'' belongs to $\operatorname{add} \mathcal{Q}_{A_l}$. We know that A is obtained from A_l (respectively $\mathcal{C}_A$ is obtained from $\mathcal{T}_{A_l}$) by a sequence of admissible operations of types (ad 1)–(ad 4) or (ad 5). In this process, the indecomposable modules from $\mathcal{Q}_{A_l}$ will become indecomposable modules of $\mathcal{Q}_A = \mathcal{Q}_{A_r}$. Further, $\mathcal{T}_{A_l}$ is a family of coray tubes. Then it follows from the description of admissible operations (ad 1)–(ad 5), presented in Section 3, that only directing modules of $\mathcal{T}_{A_l}$ may become modules of $\mathcal{Q}_A = \mathcal{Q}_{A_r}$ after application of the admissible operations leading from A_l to A, and then some admissible operations of types (ad 3) or (ad 4) must be applied. Assume $Y'' \neq 0$. Then $Y' = 0$ and $Y = Y|_{A_l} = Y''$ is an indecomposable module from Q_{A_l}. Moreover, then Y is an indecomposable module over the concealed canonical algebra C. Since $\mathcal{T}_{A_l}$ is a family of coray tubes of Γ_{A_l} separating $\mathcal{P}_{A_l} = \mathcal{P}_A$ from $\mathcal{Q}_{A_l}$, we conclude that $f : X \to Y$ factors through a module Z from $\operatorname{add} \mathcal{T}_{A_l}$. In fact, since Y is a C-module, we may assume that Z is a C-module. But then Z is a direct sum of nondirecting modules from $\mathcal{T}_{A_l}$, and so (by the above remarks) belongs to $\operatorname{add} \mathcal{C}_A$. Therefore, $f : X \to Y$ factors through a module from $\operatorname{add} \mathcal{C}_A$. Finally, assume $Y'' = 0$. Then $Y = Y|_{A_l} = Y'$ is an indecomposable directing module. Then it follows from the structure of coray tubes that $f : X \to Y$ factors through a module $Z \in \operatorname{add} \mathcal{T}_{A_l}$ without indecomposable directing direct summands. But such a module Z belongs to $\operatorname{add} \mathcal{C}_A$, and so $f : X \to Y$ has the required factorization through a module from $\operatorname{add} \mathcal{C}_A$. This proves that $\mathcal{C}_A$ satisfies (S3). Therefore, $\mathcal{C}_A$ is a separating family of almost cyclic coherent components of Γ_A, and (iii) $\Rightarrow$ (i) of Theorem A holds.

We illustrate the above considerations in the following example.

Example 4.1. Let A be the algebra given by the quiver in Fig. 1. We will show that A is a generalized multicoil enlargement of a concealed canonical algebra C. We take C as the product of $A_0 \times B_0 \times C_0$, where A_0 is the hereditary algebra of Euclidean type $\widetilde{\mathbb{D}}_6$ given by the vertices $1, 2, \ldots, 7$, B_0 is the hereditary algebra of Euclidean type $\widetilde{\mathbb{A}}_5$ given by the vertices $21, 22, \ldots, 26$, and C_0 is the hereditary algebra of Euclidean type $\widetilde{\mathbb{A}}_5$ given by the vertices $36, 37, \ldots, 41$.

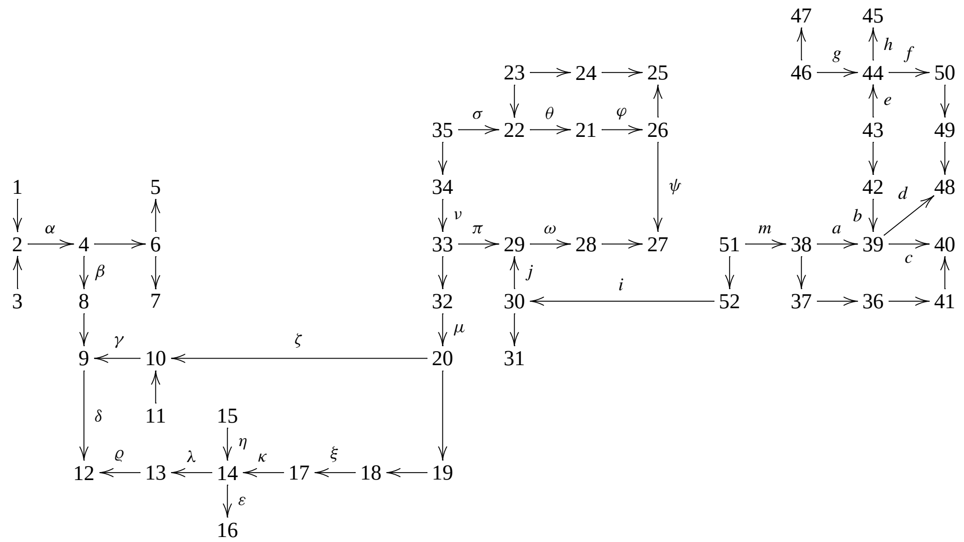


Fig. 1. A quiver bound by $\alpha\beta = 0$, $\gamma\delta = 0$, $\eta\varepsilon = 0$, $\kappa\lambda\varrho = 0$, $\zeta\gamma = 0$, $\xi\kappa\lambda = 0$, $\mu\zeta = 0$, $\nu\pi = 0$, $\pi\omega = 0$, $\sigma\theta = 0$, $\varphi\psi = 0$, $ad = 0$, $bc = 0$, $bd = 0$, $ef = 0$, $gh = 0$, $ij = 0$, $ma = 0$.

Consider the algebra A_0. Let us denote dimension-vectors

$$\mathbf{a}_1 = \begin{smallmatrix} 0 & & 0 \\ 0 & 1 & 0 \\ 0 & & 0 \end{smallmatrix}, \qquad \mathbf{a}_2 = \begin{smallmatrix} 0 & & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{smallmatrix}, \qquad \mathbf{a}_3 = \begin{smallmatrix} 0 & & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ & 1 & 0 \\ & & 0 \end{smallmatrix}, \qquad \mathbf{a}_4 = \begin{smallmatrix} 0 & & 0 & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ & 0 & 0 & \\ & & 0 & 0 \\ & 0 & 0 & 1 \end{smallmatrix}.$$

We apply (ad 1*) with pivot the simple regular A_0-module with dimension-vector $\mathbf{a}_1$, and with parameter $t = 0$. The modified algebra A_1 is given by the quiver with vertices $1, 2, \ldots, 8$ bound by $\alpha\beta = 0$. Now, we apply (ad 1*) with pivot the indecomposable A_1-module with dimension-vector $\mathbf{a}_2$, and with parameter $t = 2$. The modified algebra A_2 is given by the quiver with vertices $1, 2, \ldots, 11$ bound by $\alpha\beta = 0$. Next, we apply (ad 1*) with pivot the indecomposable A_2-module with dimension-vector $\mathbf{a}_3$, and with parameter $t = 3$. The modified algebra A_3 is given by the quiver with vertices $1, 2, \ldots, 15$ bound by $\alpha\beta = 0$, $\gamma\delta = 0$. In the next step we apply (ad 1*) with pivot the indecomposable A_3-module with dimension-vector $\mathbf{a}_4$, and with parameter $t = 0$. The modified algebra A_4 is given by the quiver with vertices $1, 2, \ldots, 16$ bound by $\alpha\beta = 0$, $\gamma\delta = 0$, $\eta\varepsilon = 0$.

Consider the algebra B_0. We will use the following dimension-vectors:

$$\mathbf{b}_1 = \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}, \qquad \mathbf{b}_2 = \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ & & \\ 0 & 0 & 0 \\ 1 & & \end{smallmatrix}, \qquad \mathbf{b}_3 = \begin{smallmatrix} 0 & & 0 & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ & 0 & 0 & \\ & & 0 & 0 \\ & 0 & 1 & 1 \\ & & & 1 \end{smallmatrix}, \qquad \mathbf{b}_4 = \begin{smallmatrix} 0 & & 0 & & \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ & 0 & 1 & & \\ & & 0 & 0 & \\ & 0 & 0 & 0 & 0 \\ & & & 0 & \end{smallmatrix}, \qquad \mathbf{b}_5 = \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ & & \\ 1 & 0 & 0 \\ 0 & & \\ 0 & & \end{smallmatrix}.$$

We apply (ad 1^*) with pivot the simple regular B_0-module with dimension-vector $\mathbf{b}_1$, and with parameter $t = 3$. The modified algebra B_1 is given by the quiver with vertices $21, 22, \ldots, 30$ bound by $\varphi\psi = 0$. Now, we apply (ad 1^*) with pivot the indecomposable B_1-module with dimension-vector $\mathbf{b}_2$, and with parameter $t = 0$. The modified algebra B_2 is given by the quiver with vertices $21, 22, \ldots, 31$ bound by $\varphi\psi = 0$. Next, we apply the admissible operation (ad 5) in four steps. The first step: we apply the operation (fad 3) with pivot the indecomposable A_4-module with dimension-vector $\mathbf{b}_3$, and with parameters $t = 3$, $s = 2$. The modified algebra A_5 is given by the quiver with vertices $1, 2, \ldots, 17$ bound by $\alpha\beta = 0$, $\gamma\delta = 0$, $\eta\varepsilon = 0$, $\kappa\lambda\varrho = 0$. The second step: we apply the operation (fad 4) with pivot the indecomposable A_5-module with dimension-vector $\mathbf{b}_4$, and with a finite sectional path consisting of the indecomposable A_5-modules with dimension-vectors

$$\begin{array}{c}0\ \ 0\\000\\000\\00\\00\\0011\\1\end{array}\rightarrow\begin{array}{c}0\ \ 0\\000\\000\\00\\00\\0011\\0\end{array}\rightarrow\begin{array}{c}0\ \ 0\\000\\000\\00\\01\\0011\\0\end{array}\rightarrow\begin{array}{c}0\ \ 0\\000\\000\\00\\00\\0001\\0\end{array}$$

and with parameters $s = 1$, $r = 2$. The modified algebra A_6 is given by the quiver with vertices $1, 2, \ldots, 20$ bound by $\alpha\beta = 0$, $\gamma\delta = 0$, $\eta\varepsilon = 0$, $\kappa\lambda\varrho = 0$, $\zeta\gamma = 0$, $\xi\kappa\lambda = 0$. The third step: we apply the operation (fad 4) with pivot the indecomposable B_2-module with dimension-vector $\mathbf{b}_5$, and with a finite sectional path consisting of the indecomposable A_6-modules with the following dimension-vectors:

$$\begin{array}{c}0\ \ 0\\000\\000\\00\quad 1\\00\\001111\\1\end{array}\rightarrow\begin{array}{c}0\ \ 0\\000\\000\\00\quad 1\\00\\001111\\0\end{array}\rightarrow\begin{array}{c}0\ \ 0\\000\\000\\00\quad 1\\01\\001111\\0\end{array}\rightarrow\begin{array}{c}0\ \ 0\\000\\000\\00\quad 1\\00\\000111\\0\end{array}\rightarrow\begin{array}{c}0\ \ 0\\000\\000\\00\quad 1\\00\\000011\\0\end{array}\rightarrow\begin{array}{c}0\ \ 0\\000\\000\\00\quad 1\\00\\000001\\0\end{array}\rightarrow\begin{array}{c}0\ \ 0\\000\\000\\00\quad 1\\00\\000000\\0\end{array}$$

and with parameters $s = 2$, $r = 1$. The modified algebra Λ_0 is given by the quiver with vertices $1, 2, \ldots, 33$ bound by $\alpha\beta = 0$, $\gamma\delta = 0$, $\eta\varepsilon = 0$, $\kappa\lambda\varrho = 0$, $\zeta\gamma = 0$, $\xi\kappa\lambda = 0$, $\varphi\psi = 0$, $\mu\zeta = 0$, $\pi\omega = 0$. The fourth step: we apply the operation (ad 4) with pivot the indecomposable Λ_0-module with dimension-vector

$$\begin{array}{ll} & 000\\ & 100\\ 0\ \ 0 & \\ 000 & 0000\\ 000 & 00\\ \ \ 00 & 00\\ \ \ \ \ 00 & \\ \ \ 000000 & \\ \ \ \ \ \ 0 & \end{array}$$

and with a finite sectional path consisting of the indecomposable Λ_0-modules with the following dimension-vectors:

$$
\begin{array}{cc} & 000\\ & 000\\ 00 & \\ 000 & 1000\\ 000 & 10\\ 00 & 10\\ 00 & \\ 001111 & \\ 1 & \end{array}
\to
\begin{array}{cc} & 000\\ & 000\\ 00 & \\ 000 & 1000\\ 000 & 10\\ 00 & 10\\ 00 & \\ 001111 & \\ 0 & \end{array}
\to
\begin{array}{cc} & 000\\ & 000\\ 00 & \\ 000 & 1000\\ 000 & 10\\ 00 & 10\\ 01 & \\ 001111 & \\ 0 & \end{array}
\to
\begin{array}{cc} & 000\\ & 000\\ 00 & \\ 000 & 1000\\ 000 & 10\\ 00 & 10\\ 00 & \\ 000111 & \\ 0 & \end{array}
\to
\begin{array}{cc} & 000\\ & 000\\ 00 & \\ 000 & 1000\\ 000 & 10\\ 00 & 10\\ 00 & \\ 000011 & \\ 0 & \end{array}
$$

$$
\to
\begin{array}{cc} & 000\\ & 000\\ 00 & \\ 000 & 1000\\ 000 & 10\\ 00 & 10\\ 00 & \\ 000001 & \\ 0 & \end{array}
\to
\begin{array}{cc} & 000\\ & 000\\ 00 & \\ 000 & 1000\\ 000 & 10\\ 00 & 10\\ 00 & \\ 000000 & \\ 0 & \end{array}
\to
\begin{array}{cc} & 000\\ & 000\\ 00 & \\ 000 & 1000\\ 000 & 10\\ 00 & 00\\ 00 & \\ 000000 & \\ 0 & \end{array}
\to
\begin{array}{cc} & 000\\ & 000\\ 00 & \\ 000 & 1000\\ 000 & 00\\ 00 & 00\\ 00 & \\ 000000 & \\ 0 & \end{array}
$$

and with parameter $r = 1$. The modified algebra Λ_1 is given by the quiver with vertices $1, 2, \dots, 35$ bound by $\alpha\beta = 0$, $\gamma\delta = 0$, $\eta\varepsilon = 0$, $\kappa\lambda\varrho = 0$, $\zeta\gamma = 0$, $\xi\kappa\lambda = 0$, $\varphi\psi = 0$, $\mu\zeta = 0$, $\pi\omega = 0$, $\nu\pi = 0$, $\sigma\theta = 0$.

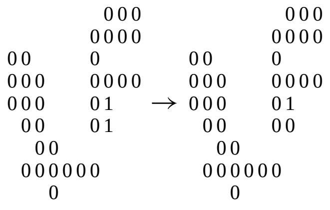
Consider the algebra C_0. Let us denote:

$$
\mathbf{c}_1 = \begin{smallmatrix} 010\\ 000 \end{smallmatrix}, \qquad
\mathbf{c}_2 = \begin{smallmatrix} 1\\ 010\\ 000 \end{smallmatrix}, \qquad
\mathbf{c}_3 = \begin{smallmatrix} 0\\ 1\\ 0\\ 0\\ 000\\ 000 \end{smallmatrix}, \qquad
\mathbf{c}_4 = \begin{smallmatrix} 00\\ 00\\ 0\\ 0\\ 010\\ 000 \end{smallmatrix}, \qquad
\mathbf{c}_5 = \begin{smallmatrix} 00\\ 000\\ 00\\ 00\\ 101\\ 111 \end{smallmatrix}.
$$

We apply (ad 1) with pivot the simple regular C_0-module with dimension-vector $\mathbf{c}_1$, and with parameter $t = 0$. The modified algebra C_1 is given by the quiver with vertices $36, 37, \dots, 42$ bound by $bc = 0$. Now, we apply (ad 1) with pivot the indecomposable C_1-module with dimension-vector $\mathbf{c}_2$, and with parameter $t = 2$. The modified algebra C_2 is given by the quiver with vertices $36, 37, \dots, 45$ bound by $bc = 0$. Next, we apply (ad 1) with pivot the indecomposable C_2-module with dimension-vector $\mathbf{c}_3$, and with parameter $t = 1$. The modified algebra C_3 is given by the quiver with vertices $36, 37, \dots, 47$ bound by $bc = 0$, $gh = 0$. Now, we apply the operation (ad 4*) with pivot the indecomposable C_3-module with dimension-vector $\mathbf{c}_4$, and with a finite sectional path consisting of the indecomposable C_3-modules with the following dimension-vectors

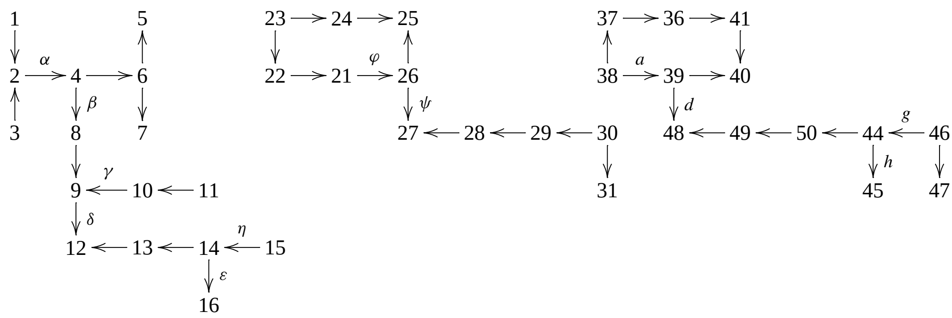
$$
\begin{smallmatrix} 00\\ 11\\ 0\\ 0\\ 000\\ 000 \end{smallmatrix}
\leftarrow
\begin{smallmatrix} 10\\ 11\\ 0\\ 0\\ 000\\ 000 \end{smallmatrix}
\leftarrow
\begin{smallmatrix} 00\\ 01\\ 0\\ 0\\ 000\\ 000 \end{smallmatrix}
\leftarrow
\begin{smallmatrix} 01\\ 01\\ 0\\ 0\\ 000\\ 000 \end{smallmatrix}
$$

and with parameter $r = 2$. The modified algebra C_4 is given by the quiver with vertices $36, 37, \dots, 50$ bound by $bc = 0$, $gh = 0$, $ad = 0$, $bd = 0$, $ef = 0$. Finally, we apply the operation (ad 4) with pivot the indecomposable C_4-module with dimension-vector $\mathbf{c}_5$ and with a finite sectional path consisting of the indecomposable Λ_1-modules with the following dimension-vectors

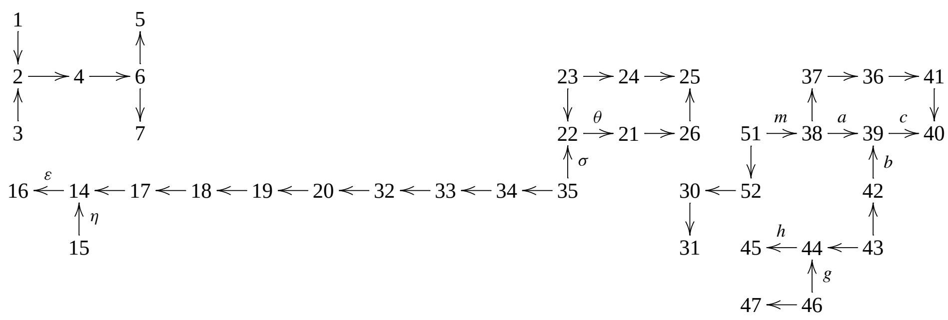


and with parameter $r = 1$. The modified algebra is equal to A.

Then the left-quasitilted algebra A_l of A is given by the quivers



bound by $\alpha\beta = 0$, $\gamma\delta = 0$, $\eta\varepsilon = 0$, $\varphi\psi = 0$, $ad = 0$, $gh = 0$. The right-quasitilted algebra A_r of A is given by the quivers



bound by $\eta\varepsilon = 0$, $\sigma\theta = 0$, $bc = 0$, $gh = 0$, $ma = 0$.

5. Proof of Theorem E

Let A be an algebra with a separating family $\mathcal{C}_A$ of almost cyclic coherent components in Γ_A, and $\operatorname{ind} A = \mathcal{P}_A \vee \mathcal{C}_A \vee \mathcal{Q}_A$. Then, by Theorem C, there are uniquely determined quasitilted algebras A_l and A_r such that $\mathcal{P}_A = \mathcal{P}_{A_l}$ and $\mathcal{Q}_A = \mathcal{Q}_{A_r}$. Moreover, A_l^{op} and A_r are almost concealed canonical algebras, and hence the indecomposable projective A-modules belong to $\mathcal{P}_A \vee \mathcal{C}_A$ while the indecomposable injective A-modules belong to $\mathcal{C}_A \vee \mathcal{Q}_A$. In particular, for any indecomposable A-module X from $\mathcal{P}_A$ we have $\mathrm{Hom}_A(D(A), \tau_A X) = 0$, and hence $\mathrm{pd}_A X \leqslant 1$, by [34, (2.5)]. Similarly, for any indecomposable A-module X from $\mathcal{Q}_A$ we have $\mathrm{Hom}_A(\tau_A^- X, A) = 0$, and hence $\mathrm{id}_A X \leqslant 1$. We will show now that $\mathrm{pd}_A X \leqslant 2$ and $\mathrm{id}_A X \leqslant 2$ for any indecomposable A-module X from $\mathcal{C}_A$. By duality, it is enough to prove the first inequality. We will apply arguments similar to those applied in the proof of [29, Proposition 1.2]. We know that Γ_{A_l} admits a separating family $\mathcal{T}_{A_l}$ of coray tubes such that $\mathcal{C}_A$ can be obtained from $\mathcal{T}_{A_l}$ by a sequence of admissible operations of types (ad 1)–(ad 5), corresponding to the admissible operations leading from A_l to A. Similarly, Γ_{A_r} admits a separating family $\mathcal{T}_{A_r}$ of ray tubes such that $\mathcal{C}_A$ can be obtained from $\mathcal{T}_{A_r}$ by a sequence of admissible operations of types (ad 1*)–(ad 5*), corresponding to the admissible operations leading from A_r to A.

Let X be a module in $\mathcal{C}_A$. We shall show that, if $\pi : P(X) \to X$ is the projective cover of X and $\Omega(X) = \ker \pi$, then $\Omega(X) = X_1 \oplus X_2$, where X_1 is projective and X_2 is a direct sum of modules lying in $\mathcal{P}_{A_l}$. Let Γ be a generalized multicoil in $\mathcal{C}_A$ containing the module X. We shall proceed by induction on the number of injective modules in Γ. Suppose first that Γ does not contain injective modules. Then Γ is a ray tube obtained from a stable tube of $\mathcal{T}_{A_l}$ by a sequence of admissible operations of type (ad 1). Since the generalized multicoils of $\mathcal{C}_A$ are pairwise orthogonal and the injective modules are in $\mathcal{C}_A \vee \mathcal{Q}_{A_r}$ we get that $\mathrm{Hom}_A(D(A), \tau_A X) = 0$. Hence $\mathrm{pd}_A X \leqslant 1$ (see [34, (2.5)]) and $\Omega(X)$ is projective. Moreover, if P' is an indecomposable direct summand of $\Omega(X)$ lying in Γ, then P' is directing. Assume now that Γ contains an injective module. Consider the exact sequence

$$0 \to \Omega(X) \to P(X) \xrightarrow{\pi} X \to 0$$

and its restriction

$$0 \to \Omega(X)|_{A_r} \to P(X)|_{A_r} \xrightarrow{\pi|_{A_r}} X|_{A_r} \to 0$$

to the full convex subcategory A_r. Clearly, $P(X)|_{A_r}$ is the projective cover of $X|_{A_r}$ in $\mathrm{mod}\, A_r$. From the proof of Theorem C and [4, (4.1)], we have that $X|_{A_r}$ is a direct sum of modules lying in $\mathcal{T}_r$. Since $\mathcal{T}_r$ has no injective modules, we get by the above remarks that $\Omega(X|_{A_r}) = \Omega(X)|_{A_r}$ is projective and any indecomposable direct summand of $\Omega(X)|_{A_r}$ lying in $\mathcal{T}_r$ is directing. Moreover, observe that, if an indecomposable projective directing module P' in $\mathcal{T}_r$ occurs in the support of $\mathrm{Hom}_{A_r}(-, M|_{A_r})$ for a copivot M of one of the admissible operations of type (ad 1*)–(ad 5*), then the indecomposable projective A-module P'' with top $P''/\mathrm{rad}\, P'' \cong P'/\mathrm{rad}\, P'$ lies in $\mathcal{P}_{A_l}$ (see [4], Section 2 and Lemmas 3.8, 3.9). We know also from Theorem C that if X is an indecomposable A-module such that $X|_{A_r}$ is a direct sum of modules in $\mathcal{P}_{A_r}$, then X lies in $\mathcal{P}_A$. Observe that our module $P(X)$ (respectively $\Omega(X)$) is obtained from $P(X)|_{A_r}$ (respectively $\Omega(X)|_{A_r}$) by means of the sequence of coextensions leading from A_r to A and adding all indecomposable direct summands whose restrictions to A_r are zero. Hence we conclude that $\Omega(X) = X_1 \oplus X_2$

where X_1 is projective and X_2 is a direct sum of modules from $\mathcal{P}_A$. This proves the required inequality $\operatorname{pd}_A X \leqslant 2$.

Now we know that $\operatorname{pd}_A X \leqslant 2$ for any module X in $\mathcal{P}_{A_l} \vee \mathcal{C}_A$ and $\operatorname{id}_A Y \leqslant 2$ for any module Y in $\mathcal{C}_A \vee \mathcal{P}_{A_r}$. Moreover, for any module Z, the projective cover $P(Z)$ belongs to $\operatorname{add}(\mathcal{P}_{A_l} \vee \mathcal{C}_A)$, and hence also $\Omega(Z)$ belongs to $\operatorname{add}(\mathcal{P}_{A_l} \vee \mathcal{C}_A)$. Therefore, $\operatorname{pd}_A \Omega(Z) \leqslant 2$ and $\operatorname{pd}_B Z \leqslant 3$, so $\operatorname{gl\,dim} A \leqslant 3$.

References

[1] M. Auslander, I. Reiten, S.O. Smalø, Representation Theory of Artin Algebras, Cambridge Stud. Adv. Math., vol. 36, Cambridge Univ. Press, 1995.

[2] I. Assem, A. Skowroński, Minimal representation-infinite coil algebras, Manuscripta Math. 67 (1990) 305–331.

[3] I. Assem, A. Skowroński, Multicoil algebras, in: Representations of Algebras, in: CMS Conf. Proc., vol. 14, Amer. Math. Soc., Providence, RI, 1993, pp. 29–68.

[4] I. Assem, A. Skowroński, B. Tomé, Coil enlargements of algebras, Tsukuba J. Math. 19 (1995) 453–479.

[5] K. Bongartz, Algebras and quadratic forms, J. London. Math. Soc. 28 (1983) 461–469.

[6] F.U. Coelho, A. Skowroński, On the Auslander–Reiten components for quasitilted algebras, Fund. Math. 149 (1996) 67–82.

[7] W. Crawley-Boevey, Regular modules over tame hereditary algebras, Proc. London Math. Soc. 62 (1991) 490–508.

[8] W. Crawley-Boevey, Modules of finite length over their endomorphism rings, in: Representations of Algebras and Related Topics, in: London Math. Soc. Lecture Note Ser., vol. 168, Cambridge Univ. Press, 1992, pp. 127–184.

[9] G. D'Este, C.M. Ringel, Coherent tubes, J. Algebra 87 (1984) 150–201.

[10] V. Dlab, C.M. Ringel, Indecomposable Representations of Graphs and Algebras, Mem. Amer. Math. Soc., vol. 173, Amer. Math. Soc., Providence, RI, 1976.

[11] V. Dlab, C.M. Ringel, The representations of tame hereditary algebras, in: Representation Theory of Algebras, in: Lecture Notes in Pure and Appl. Math., vol. 37, Dekker, New York, 1978, pp. 329–353.

[12] Y.A. Drozd, Tame and wild matrix problems, in: Representation Theory II, in: Lecture Notes in Math., vol. 832, Springer, Berlin, 1980, pp. 242–258.

[13] O. Enge, Quasitilted algebras have preprojective components, Colloq. Math. 83 (2000) 55–69.

[14] W. Geigle, H. Lenzing, A class of weighted projective curves arising in representation theory of finite-dimensional algebras, in: Singularities, Representations of Algebras, and Vector Bundles, in: Lecture Notes in Math., vol. 1273, Springer, Berlin, 1987, pp. 265–297.

[15] D. Happel, I. Reiten, S.O. Smalø, Tilting in Abelian Categories and Quasitilted Algebras, Mem. Amer. Math. Soc., vol. 575, Amer. Math. Soc., Providence, RI, 1996.

[16] D. Happel, C.M. Ringel, Tilted algebras, Trans. Amer. Math. Soc. 274 (1982) 399–443.

[17] D. Happel, I. Reiten, Hereditary abelian categories with tilting object over arbitrary base fields, J. Algebra 256 (2002) 414–432.

[18] H. Lenzing, H. Meltzer, Sheaves on a weighted projective line of genus one, and representations of a tubular algebra, in: Representations of Algebras, in: CMS Conf. Proc., vol. 14, Amer. Math. Soc., Providence, RI, 1993, pp. 313–337.

[19] H. Lenzing, H. Meltzer, Tilting sheaves and concealed-canonical algebras, in: Representations of Algebras, in: CMS Conf. Proc., vol. 18, Amer. Math. Soc., Providence, RI, 1996, pp. 455–473.

[20] H. Lenzing, J.A. de la Peña, Wild canonical algebras, Math. Z. 224 (1997) 403–425.

[21] H. Lenzing, J.A. de la Peña, Concealed-canonical algebras and separating tubular families, Proc. London Math. Soc. 78 (1999) 513–540.

[22] H. Lenzing, A. Skowroński, Quasi-tilted algebras of canonical type, Colloq. Math. 71 (1996) 161–181.

[23] S. Liu, Tilted algebras and generalized standard Auslander–Reiten components, Arch. Math. (Basel) 61 (1993) 12–19.

[24] P. Malicki, Generalized coil enlargements of algebras, Colloq. Math. 76 (1998) 57–83.

[25] P. Malicki, A. Skowroński, Almost cyclic coherent components of an Auslander–Reiten quiver, J. Algebra 229 (2000) 695–749.

[26] P. Malicki, A. Skowroński, B. Tomé, Indecomposable modules in coils, Colloq. Math. 93 (2002) 67–130.

[27] H. Meltzer, Auslander–Reiten components for concealed-canonical algebras, Colloq. Math. 71 (1996) 183–202.

[28] J.A. de la Peña, On the dimension of the module-varieties of tame and wild algebras, Comm. Algebra 19 (1991) 1795–1807.

[29] J.A. de la Peña, A. Skowroński, Geometric and homological characterizations of polynomial growth strongly simply connected algebras, Invent. Math. 126 (1996) 287–296.

[30] I. Reiten, A. Skowroński, Sincere stable tubes, J. Algebra 232 (2000) 64–75.

[31] I. Reiten, A. Skowroński, Characterizations of algebras with small homological dimensions, Adv. Math. 179 (2003) 122–154.

[32] I. Reiten, A. Skowroński, Generalized double tilted algebras, J. Math. Soc. Japan 56 (2004) 269–288.

[33] C.M. Ringel, Representations of K-species and bimodules, J. Algebra 41 (1976) 269–302.

[34] C.M. Ringel, Tame algebras and integral quadratic forms, in: Lecture Notes in Math., vol. 1099, Springer, Berlin, 1984.

[35] C.M. Ringel, The canonical algebras, with an appendix by W. Crawley-Boevey, in: Topics in Algebra, in: Banach Center Publ., vol. 26, PWN, Warszawa, 1990, pp. 407–432.

[36] A. Skowroński, Algebras of polynomial growth, in: Topics in Algebra, in: Banach Center Publ., vol. 26, PWN, Warszawa, 1990, pp. 535–568.

[37] A. Skowroński, Generalized standard Auslander–Reiten components without oriented cycles, Osaka J. Math. 30 (1993) 515–527.

[38] A. Skowroński, Generalized standard Auslander–Reiten components, J. Math. Soc. Japan 46 (1994) 517–543.

[39] A. Skowroński, On the composition factors of periodic modules, J. London Math. Soc. 49 (1994) 477–492.

[40] A. Skowroński, On omnipresent tubular families of modules, in: Representations of Algebras, in: CMS Conf. Proc., vol. 18, Amer. Math. Soc., Providence, RI, 1996, pp. 641–657.

[41] A. Skowroński, Tame quasi-tilted algebras, J. Algebra 203 (1998) 470–480.

[42] A. Skowroński, On the structure of periodic modules over tame algebras, Proc. Amer. Math. Soc. 127 (1999) 1941–1949.

[43] A. Skowroński, Generalized canonical algebras and standard stable tubes, Colloq. Math. 90 (2001) 77–93.

[44] A. Skowroński, On Artin algebras with almost all indecomposable modules of projective or injective dimension at most one, Centr. European J. Math. 1 (2003) 108–122.